The sequence of numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, and so on, are known as the Fibonacci numbers. What does “so on” actually mean? Well, it means that we start with 0 and 1 and keep getting the next number by adding the two previous numbers. More formally:

\[ f_n = \begin{cases} 
0 & \text{if } n = 0, \\
1 & \text{if } n = 1, \\
f_{n-1} + f_{n-2} & \text{if } n \geq 2.
\end{cases} \]

Fibonacci numbers pop up in various areas in science—which we won’t bother going into here. With one exception: the powers of the matrix

\[ M = \begin{pmatrix} 1 & 1 \\
1 & 0 \end{pmatrix}. \]

We could have written:

\[ M = \begin{pmatrix} f_2 & f_1 \\
 f_1 & f_0 \end{pmatrix}. \]

That’s trivial, it gets more powerful when, well, you look at powers of \( M \). It is the case that

\[ M^n = \begin{pmatrix} f_{n+1} & f_n \\
f_n & f_{n-1} \end{pmatrix} \]

for all \( n \). That’s pretty easy to see:

\[
\begin{align*}
M^{n+1} &= M^n M \\
&= \begin{pmatrix} f_{n+1} & f_n \\
f_n & f_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\
1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f_{n+1} + f_n & f_{n+1} \\
f_{n+1} & f_n \end{pmatrix} \\
&= \begin{pmatrix} f_{n+2} & f_{n+1} \\
f_{n+1} & f_n \end{pmatrix}
\end{align*}
\]

and easily verified experimentally by calculating \( M^n \) for some values of \( n \).
Now let’s see how $M$ is related to the golden ratio, which is $$\varphi = (1 + \sqrt{5})/2.$$ Let’s determine the eigenvalues of $M$. We need to ask: For what values of $\lambda$ are there non-zero vectors $x$ so that $Mx = \lambda x$? As we saw, that question is the same as asking whether $Mx = \lambdaIx$, which is equivalent to $(M - \lambda I)x = 0$. We are asking whether $(M - \lambda I)x = 0$ has a solution $x \neq 0$. We know that this is precisely the case if $M - \lambda I$ is singular (non-invertible). Which is equivalent to $\det(M - \lambda I) = 0$.

So let us calculate $\det(M - \lambda I)$. That’s

$$\det(M - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (1 - \lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1.$$  

This last expression, is just the characteristic polynomial of $M$—which in general is defined as $\det(\lambda I - M)$. So for $\lambda$ to be an eigenvalue of $M$ we need that $\lambda^2 - \lambda - 1 = 0$. This is equivalent to $(\lambda - 1/2)^2 = 5/4$, so either $\lambda = (1 + \sqrt{5})/2$, the golden ratio, or $\varphi$ as we called it, or $\lambda = (1 - \sqrt{5})/2$, or $\psi$, as it is sometimes called. There are some interesting relationships between $\varphi$ and $\psi$, we’ll need the following two in particular:

$$1 - \psi = 1 - (1 - \sqrt{5})/2 = (1 + \sqrt{5})/2 = \varphi,$$

and

$$1/\varphi = \frac{2}{1 + \sqrt{5}} = \frac{2(1 - \sqrt{5})}{(1 + \sqrt{5})(1 - \sqrt{5})} = \frac{2(1 - \sqrt{5})}{1 - 5} = (\sqrt{5} - 1)/2 = -\psi.$$  

To summarize: $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has two eigenvalues: $\varphi = (1 + \sqrt{5})/2$, and $\psi = (1 - \sqrt{5})/2$. What about the corresponding eigenvectors? For $\varphi$ we need a vector $v_\varphi$ with $Mv_\varphi = \varphi v$, and for $\psi$ we need $v_\psi$ with $Mv_\psi = \psi v$. We can find these vectors by solving the linear system $Mx = \lambda x$ for $x$, but in this case, it turns out the eigenvectors are pretty easy to guess: $v_\varphi = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$, and
\(v_\psi = \begin{pmatrix} 1 \\ -\varphi \end{pmatrix};\)

\[
Mv_\psi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ 1 \end{pmatrix} = \begin{pmatrix} \varphi + 1 \\ \varphi \end{pmatrix} = \varphi \begin{pmatrix} (\varphi + 1)/\varphi \\ 1 \end{pmatrix} = \varphi \begin{pmatrix} 1 + 1/\varphi \\ 1 \end{pmatrix} = \varphi \begin{pmatrix} 1 - \psi \\ 1 \end{pmatrix} = \varphi \varphi \varphi v_\psi.
\]

Verifying that \(v_\psi\) is the eigenvector belonging to eigenvalue \(\psi\) is similar. So if we let

\[
P = (v_\varphi | v_\psi) = \begin{pmatrix} \varphi & 1 \\ 1 & -\varphi \end{pmatrix},
\]

and

\[
D = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix},
\]

then what we just showed is that \(MP = PD\), or, since \(P\) is invertible, \(P^{-1}MP = D\), showing that \(M\) is diagonalizable, and \(M = PDP^{-1}\). And, by the way, it’s pretty easy to get \(P^{-1}\) explicitly using the special rule for \(2 \times 2\) matrices:

\[
P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -\varphi & -1 \\ -1 & \varphi \end{pmatrix} = (-\varphi^2 - 1) \begin{pmatrix} -\varphi & -1 \\ -1 & \varphi \end{pmatrix} = (1 + \varphi^2)P.
\]

Nearly there. For the final step, we write

\[
M^n = (P^{-1}DP)^n
\]

\[
= (P^{-1}DP)(P^{-1}DP)\ldots(P^{-1}DP)
\]

\[
= (P^{-1}D)(PP^{-1}DP\ldotsP^{-1}DP)
\]

\[
= P^{-1}D^n P.
\]

So to get \(f_{n+1}\), which is in the upper-left-hand corner of \(M^n\), all we need to do is calculate the upper-left-hand corner of \(P^{-1}D^n P\) which is much easier, and
which we can do explicitly:

\[
P^{-1}DnP = P^{-1} \begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix} P
\]

\[
= \frac{1}{1 + \varphi^2} \begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix} P
\]

\[
= \frac{1}{1 + \varphi^2} \begin{pmatrix} -\varphi & -1 \\ -1 & \varphi \end{pmatrix} \begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix} \begin{pmatrix} -\varphi & -1 \\ -1 & \varphi \end{pmatrix}
\]

\[
= \frac{1}{1 + \varphi^2} \begin{pmatrix} \varphi^{n+1} & \psi^n \\ \varphi^n & -\varphi \psi^n \end{pmatrix} \begin{pmatrix} -\varphi & -1 \\ -1 & \varphi \end{pmatrix}
\]

\[
= \frac{1}{1 + \varphi^2} \begin{pmatrix} \varphi^{n+2} + \psi^n \\ \varphi^{n+1} + \psi^{n-1} \end{pmatrix} \begin{pmatrix} \varphi^{n+1} - \varphi \psi^n \\ \varphi^n - \varphi \psi^{n-1} \end{pmatrix}
\]

We could keep simplifying the whole matrix, but let’s just concentrate on the upper left-hand corner:

\[
\varphi^{n+2} + \psi^n = \varphi(\varphi^{n+1} + \psi^n/\varphi) = \varphi(\varphi^{n+1} - \psi^{n+1}).
\]

And similarly, for the factor \(1 + \varphi^2 = \varphi(1/\varphi + \varphi) = \varphi(\varphi - \psi)\). So we conclude that

\[
f_{n+1} = \frac{\varphi^{n+2} + \psi^n}{1 + \varphi^2} = \frac{\varphi(\varphi^{n+1} - \psi^{n+1})}{\varphi(\varphi - \psi)} = \frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi},
\]

or, restated for the n-th Fibonacci number:

\[
f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}.
\]

And that was our goal. A closed formula relating the Fibonacci numbers to the golden ratio.