Removing Independently Even Crossings*

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Abstract

We show that $\mathrm{cr}(G) \leq \binom{2\,\mathrm{iocr}(G)}{2}$ settling an open problem of Pach and Tóth [4, 1]. Moreover, $\mathrm{iocr}(G) = \mathrm{cr}(G)$ if $\mathrm{iocr}(G) \leq 2$.

1 Crossing Numbers

Pach and Tóth point out in "Which Crossing Number is It Anyway?" that there have been many different ideas on how to define a notion of crossing number including—using current terminology—the following (see [5, 13]):

crossing number: $\mathrm{cr}(G)$, the smallest number of crossings in a drawing of G,

pair crossing number:[1] $\mathrm{pcr}(G)$, the smallest number of pairs of edges crossing in a drawing of G,

odd crossing number: $\mathrm{ocr}(G)$, the smallest number of pairs of edges crossing oddly in a drawing of G.

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We make the typical assumptions on drawings of a graph: there are only finitely many crossings, no more than two edges cross in a point, edges do not pass through vertices, and edges do not touch. (For a detailed discussion see [13].) One may consider relaxing some of these assumptions. For example, allowing more than two edges to cross in a point leads to the notion of degenerate crossing number introduced by Pach and Tóth [6]. Relaxing the touching condition has no effect on crossing number or pair crossing number, but it would make odd crossing number identical to zero. There are also conditions one might consider adding, such as requiring edges to be straight-line segments, which leads to the notion of rectilinear crossing number (for which the pair and odd versions coincide with the standard version). Finally, there is the issue of whether adjacent edges are allowed to cross or whether their crossings should count. Tutte [17] wrote

> "We are taking the view that crossings of adjacent edges are trivial, and easily got rid of."

While this is true for the standard crossing number, it is not at all obvious for other variants (or the particular variant that Tutte was studying). Székely [13] comments "We interpret this sentence as a philosophical view and not a mathematical claim."

In [4], Pach and Tóth suggest a systematic study of this issue (see also [1, Section 9.4]): they introduce two rules that can be applied to any notion of crossing number. "Rule +" restricts the drawings to drawings in which adjacent edges are not allowed to cross. "Rule −" allows crossings of adjacent edges, but does not count them towards the crossing number. Pairing ocr, pcr, and cr with any of these two rules gives a total of eight possible variants (since $\mathrm{cr}_+ = \mathrm{cr}$ as we mentioned above); one of them has its own name: $\mathrm{iocr} := \mathrm{ocr}_-$, the *independent odd crossing number*, introduced by Tutte. The figure below is based on a similar figure from [1].

Rule +	ocr_+	pcr_+	cr
	ocr	pcr	
Rule −	$\mathrm{iocr} = \mathrm{ocr}_-$	pcr_-	cr_-

Very little is known about the relationships between these crossing number variants, apart from what immediately follows from the definitions: the values in the display increase monotonically as one moves from the left to the right and from the bottom to the top. Even the question $\mathrm{cr} = \mathrm{cr}_-$

[1]Recently, the book by Tao and Vu [15] on additive combinatorics defined the crossing number as pcr.

remains open. Pach and Tóth did show that $\mathrm{cr}(G) \leq \binom{2\,\mathrm{ocr}(G)}{2}$, and this implies that five of the variants, namely $\mathrm{ocr}_+, \mathrm{ocr}, \mathrm{pcr}_+, \mathrm{pcr}$, and cr cannot be arbitrarily far apart, but the result does not cover the "Rule $-$" variants. For cr versus ocr, the bound by Pach and Tóth is still the best known, though it is expected to be far from the truth. It implies that $\mathrm{cr}(G) \leq \binom{2\,\mathrm{pcr}(G)}{2}$, a bound that can be strengthened: Valtr [18] showed that $\mathrm{cr}(G) = O(\mathrm{pcr}^2(G)/\log \mathrm{pcr}(G))$, which Tóth [16] improved to $\mathrm{cr}(G) = O(\mathrm{pcr}^2(G)/\log^2 \mathrm{pcr}(G))$. Again, these bounds are not expected to be optimal, and, indeed, cr = pcr has been conjectured. On the other hand, we do know that ocr and pcr (and, therefore, ocr and cr) differ: the authors showed that there is an infinite family of graphs with $\mathrm{ocr}(G) < 0.867 \cdot \mathrm{pcr}(G)$ [12]. This separation was improved by Tóth to $\mathrm{ocr}(G) < 0.855 \cdot \mathrm{pcr}(G)$ [16]. The upper bound $\mathrm{pcr}(G) \leq \binom{2\,\mathrm{ocr}(G)}{2}$ which follows from the bound by Pach and Tóth is still the best known in this case as well.

In this paper, we show that all eight crossing number variants are within a square of each other:

Theorem 1.1. $\mathrm{cr}(G) \leq \binom{2\,\mathrm{iocr}(G)}{2}$.

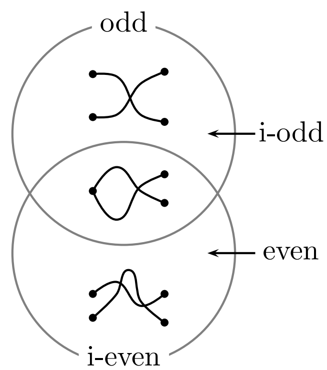
This answers an open problem from [4, Problem 13]; also see [1, Problem 9.4.7]. Pach and Tóth asked whether there are functions f, g, h for which $\mathrm{cr}(G) \leq f(\mathrm{cr}_-(G))$, $\mathrm{pcr}(G) \leq g(\mathrm{pcr}_-(G))$, and $\mathrm{ocr}(G) \leq h(\mathrm{iocr}(G))$ for all graphs G. Theorem 1.1 implies that $f = g = h = \binom{2x}{2}$ will do, but this is probably not the optimal choice for f, g, and h, and quite possibly not for bounding cr in terms of iocr either.

Theorem 1.1 immediately implies that $\mathrm{iocr}(G) = \mathrm{cr}(G)$ if $\mathrm{iocr}(G) \leq 1$. In Section 3 we strengthen this to $\mathrm{iocr}(G) = \mathrm{cr}(G)$ if $\mathrm{iocr}(G) \leq 2$. Previously we showed that $\mathrm{ocr}(G) = \mathrm{cr}(G)$ if $\mathrm{ocr}(G) \leq 3$ [9], but the result for iocr is harder, since a bound on $\mathrm{iocr}(G)$ does not imply any a priori bound on the number of edges crossing some other edge oddly. Indeed, the new result generalizes the Hanani-Tutte theorem, which states that $\mathrm{iocr}(G) = 0$ implies that $\mathrm{cr}(G) = 0$. There are aspects of the Hanani-Tutte theorem which are still not well understood, for example to what extent it relies on the underlying surface: it has only recently been extended to the projective plane, that is, we now know that $\mathrm{iocr}_{N_1}(G) = \mathrm{cr}_{N_1}(G)$ if $\mathrm{iocr}_{N_1}(G) = 0$ [8]. However, it is not clear, how to extend this to the case that $\mathrm{iocr}_{N_1}(G) \leq 1$ or how to prove the Hanani-Tutte theorem for surfaces beyond the projective plane. We do know that $\mathrm{ocr}_S(G) = \mathrm{cr}_S(G)$ if $\mathrm{ocr}_S(G) \leq 2$ for arbitrary surfaces S [10].

The independent odd crossing number is implicit in Tutte's paper "Toward a Theory of Crossing Number" which attempts to build an algebraic foundation for the study of the standard crossing number [17]. From an algebraic point of view, ocr and iocr are much more convenient parameters than the standard crossing number; for example, as Pach and Tóth pointed out, ocr $\leq k$ and iocr $\leq k$ can be recast as problems over vector-spaces [5]. Tutte's algebraic approach has been continued by Székely [13, 14] and, along different lines, Norine [2] and van der Holst [19]. Theorem 1.1 justifies the approach of studying standard crossing number via independent odd crossing number, by showing that they are not too far apart; indeed, it is tempting to conjecture that $\mathrm{cr}(G) = O(\mathrm{iocr}(G))$. And in spite of the fact that determining the independent odd crossing number of a graph is **NP**-complete [7], we feel that due to its algebraic nature it offers an intriguing and underutilized alternative approach to algorithmic aspects of crossing number problems.

2 Removing Even More Crossings

An edge in a drawing of a graph is *odd* if it is part of an *odd pair*, which is a pair of edges that cross an odd number of times. Edges that are not odd are *even*, and they cross every edge an even number of times (possibly zero times). An edge in a drawing is *independently odd* if it is part of an *independently odd pair*, which is a pair of non-adjacent edges that cross an odd number of times. Edges that are not independently odd are *independently even*; thus, an independently even edge crosses all non-adjacent edges evenly (possibly zero times), while it may cross adjacent edges arbitrarily. For convenience, we will usually write *i-odd* for independently odd and *i-even* for independently even. Throughout this paper graphs are simple, that is, they have no loops or multiple edges, unless we say otherwise.



Pach and Tóth showed that if E is the set of even edges in a drawing D of G, then G can be redrawn so that all edges in E are crossing-free. As a corollary, they obtained $\mathrm{cr}(G) \leq \binom{2\,\mathrm{ocr}(G)}{2}$ [5]. We strengthen the Pach-Tóth result to the case that E is the set of independently even edges. According to Pach and Sharir [3], this has been conjectured.

Our redrawing has the additional property that for every crossing-free cycle C, the edges and vertices of the graph in the interior (exterior) of

C remain in the interior (exterior) of C after redrawing; we call such a redrawing *stable.*

Lemma 2.1. *If D is a drawing of a graph G in the plane, then G has a stable redrawing in which the independently even edges of D are crossing-free and every pair of edges crosses at most once.*

With this lemma, the proof of Theorem 1.1 is immediate.

Proof of Theorem 1.1. Start with a drawing D of G that realizes $\text{iocr}(G)$, that is, $\text{iocr}(D) = \text{iocr}(G)$. If F is the set of i-odd edges in D, then $|F| \leq 2\,\text{iocr}(D)$. By Lemma 2.1, there is a drawing of G with at most $\binom{|F|}{2}$ crossings. □

To prove Lemma 2.1, we adapt the following result (a different strengthening of the Pach-Tóth result) from odd edges to i-odd edges. The *rotation* of a vertex is the cyclic order in which edges leave the vertex in a drawing, read clockwise. The *rotation system* of a drawing is the collection of all vertex rotations.

Lemma 2.2 (Pelsmajer, Schaefer, and Štefankovič [9])**.** *If D is a drawing of G in the plane and F is the set of odd edges in D, then G has a redrawing with the same rotation system, in which $G - F$ is crossing-free and there are no new pairs of edges that cross an odd number of times.*

Remark 1. We may assume that the redrawing in Lemma 2.2 is stable; if G is connected, this is a consequence of Lemma 2.2 not changing the rotation: a vertex v lying inside (or outside) a crossing-free cycle C of G in D, remains on the same side of the drawing, since a path from v to C has to start on the same side of C after the redrawing (since the rotation system does not change) and C remains crossing-free (since it belongs to $G - F$), so the path cannot cross C. If G is not connected, we can find a stable redrawing of each connected component of G and combine them into a single stable redrawing, using the method of the proof of Claim 1 in Section 3.

Splitting a vertex means creating two copies of the vertex with an edge between them so that any edge incident to the original vertex is incident to exactly one of the two copies. (According to this definition, it makes sense to talk about the edges of the original graph occurring in the graph after a vertex split, even though the incidences of edges will change.)

Now we can state our analogue of Lemma 2.2 for i-odd edges.

Lemma 2.3. *If D is a drawing of G in the plane, and F is the set of i-odd edges in D, then one can apply a sequence of vertex splits to obtain a graph G' with drawing D' and the set F' of i-odd edges in D', such that (1) there are no new independent odd pairs (so $F' \subseteq F$), (2) every edge of $G' - F'$ that is not a cut-edge of $G' - F'$ is crossing-free in D', and (3) if C is a cycle of $G' - F'$ and $v \in V(C)$, then v has at most one incident edge in the interior of C, and at most one incident edge in the exterior of C.*

An edge is a cut-edge if and only if it belongs to no cycles, so Property (2) can be restated as saying that the union of cycles in $G' - F'$ is crossing-free in D'. Also, if Y is the set of cut-edges of $G' - F'$, then $G' - (F' \cup Y)$ is crossing-free in D'.

Proof of Lemma 2.3. Fix a drawing D of $G = (V, E)$ and let F be the set of i-odd edges in D. We establish the theorem by induction. We will modify G during the proof by splitting vertices, namely a vertex of degree d is split into two vertices of degrees $d_1, d_2 \geq 3$. We have $d_1 + d_2 = d + 2$, so $d_1^3 + d_2^3 < d^3$ and thus we can use induction over the *weight*

$$w(G) := \sum_{v \in V} d(v)^3$$

of G where $d(v)$ is the degree of v in G. For two graphs of the same weight, we induct over the number of cycles that are not crossing-free.

Suppose that C is a crossing-free cycle, with a vertex u that is incident to more than one edge on the same side of C. We modify the graph by splitting u into u_1 (replacing u on C) and u_2 (attached to the edges on the side with more than one edge) and inserting an edge between u_1 and u_2. This operation results in a graph G' with smaller weight and it does not create new i-odd edges (since edges in the exterior of C cannot cross edges on the interior, as all edges along C are crossing-free). We can now apply induction to G' to obtain the result. Thus, we may assume that for every vertex u in a crossing-free cycle C, u is incident to at most one edge on the interior of C and at most one edge on the exterior of C. It follows that any two edges incident to a vertex u in a crossing-free cycle do not cross.

Suppose that C is a cycle made up of i-even edges only, and C is not crossing-free. At each vertex u of C we can ensure that the two edges of C incident at u (say, e and f) cross evenly by modifying the rotation at u and redrawing G close to u (Figure 1). The rotation of the remaining edges at u can then be changed so that each of them crosses e and f evenly (Figure 2). After the redrawing, all the edges of C are even and we can apply Lemma 2.2 to remove all crossings with edges of C without changing the rotation system or adding new pairs of edges that cross oddly. Now C is crossing-free, and no new i-odd pairs have been added. Suppose that C' is a cycle that was crossing-free before the redrawing. If C and C' share a vertex u, then the rotation at u is not modified when making C crossing-free, so the drawing of C' near u is unchanged. C' remains crossing-free under the stable redrawing of Lemma 2.2, too. Thus we have decreased the number of cycles that are not crossing-free.

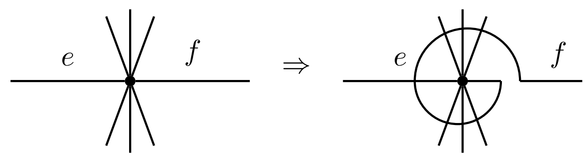


Figure 1: If e and f form an odd pair, redraw near u.

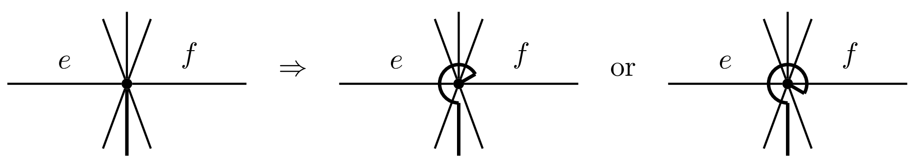


Figure 2: If another edge incident to u crosses e oddly and f evenly (or vice-versa), or if it crosses both e and f oddly, it can be redrawn so that it crosses both e and f evenly.

We can therefore assume that any cycle consisting of i-even edges is crossing-free. Any other i-even edge is a cut-edge in the graph restricted to i-even edges. □

With Lemma 2.3, we can now prove Lemma 2.1. Note that Property (3) is not needed for this proof; it is used in Section 3.

Proof of Lemma 2.1. Fix a drawing D of G and let F be the set of i-odd edges in D. Apply Lemma 2.3 to obtain a graph G' with drawing D', let F' be the set of i-odd edges in D', and let Y be the set of cut-edges in $G' - F'$. Since $F' \cup Y$ contains all crossings in D', $G' - (F' \cup Y)$ is crossing-free in D' and we can let S be the set of its faces. Within each face of S, the edges of Y contained in it can be redrawn one-by-one without creating any crossings, since no edge of Y can complete a path that cuts a face in two (because then it would be part of a cycle in $G' - F'$, which contradicts it being a cut-edge of $G' - F'$). This yields a crossing-free drawing of $G' - F'$, and each of its faces corresponds to a face of S, with boundary formed from the boundary of the face of S and the edges of Y in that face. Therefore, each edge of F' still has both endpoints incident to a face. Within each face, all such edges of F' can be drawn so that every pair of edges crosses at most once.[2]

Since G' was obtained from G by a sequence of vertex splits, G can be obtained from G' by a sequence of edge contractions. The edges in $E(G') - E(G)$ are crossing-free, so applying that sequence of contractions to the current drawing of G' yields a drawing of G in which $G - F'$ is crossing-free and each pair of edges in F' crosses at most once. Since $F' \subseteq F$, it only remains to show that the overall redrawing is stable.

Let C be any cycle in $G - F$ that is crossing-free in D. If a vertex u of C is split by Lemma 2.3, the cycle is either lengthened by one as u is replaced by an edge and its endpoints u_1 and u_2, or u's position in C is merely replaced by u_1 or u_2. In this way, C is replaced by a crossing-free cycle C'. Vertices and edges on the interior (exterior) of C end up in the interior (exterior) of C', and if the split edge is contracted, they end up in the interior (exterior) of C again. This property also holds true when splitting and contracting a vertex that is not in C. Other redrawings performed in the proof of Lemma 2.3 do not affect the drawing near C except when Lemma 2.2 is applied, and redrawings from Lemma 2.2 are stable. Finally, when we redraw edges of $F' \cup Y$ at the beginning of this proof, we do not switch between the interior and exterior of C. So overall, our redrawing is stable. □

3 Small Independent Odd Crossings Numbers

Theorem 3.1. *If* $\mathrm{iocr}(G) \leq 2$*, then* $\mathrm{cr}(G) = \mathrm{iocr}(G)$*.*

The proof is based on an analysis of the "odd configurations" that can occur in a drawing; we performed such an analysis when we proved $\mathrm{ocr}(G) \leq 3$ implies $\mathrm{cr}(G) = \mathrm{ocr}(G)$ in [9]. The present situation is more difficult. In [9], we used Lemma 2.2 and then contracted all crossing-free edges, reducing the problem to a multigraph with fixed rotation in which every edge contributes to ocr. Since $\mathrm{ocr} \leq 3$, we then only had to analyze very small multigraphs.

[2]We will see this redrawing technique again in the next section: redrawing cut-edges of $G' - F'$, which preserves faces to the extent that each edge of F' continues to have a face that is incident to both its endpoints.

The same method will not work here because contractions affect iocr, and ocr is unknown. Here, we only have Lemma 2.3.

Proof. Fix a drawing D of G realizing $\mathrm{iocr}(G)$ and let $F \subseteq E(G)$ be the set of i-odd edges in D. Let G' with drawing D' be as in Lemma 2.3, with $F' \subseteq F$ the set of i-odd edges in D'. Then $\mathrm{iocr}(D') \leq \mathrm{iocr}(D) = \mathrm{iocr}(G)$.

Suppose that G' can be redrawn with at most $\mathrm{iocr}(D')$ crossings, so that $G' - F'$ is crossing-free. We can then obtain a drawing of G with at most $\mathrm{iocr}(D')$ crossings by contracting the edges of $E(G') - E(G)$, which are all crossing-free. Thus, $\mathrm{cr}(G) \leq \mathrm{iocr}(D')$. Since $\mathrm{iocr}(D') \leq \mathrm{iocr}(G)$ and $\mathrm{iocr}(G) \leq \mathrm{cr}(G)$ by their definitions, this yields $\mathrm{cr}(G) = \mathrm{iocr}(G)$.

By the argument in the preceding paragraph we can prove the theorem by establishing the following claim.

> **Main Claim** Suppose that G is a graph with a drawing D for which $\mathrm{iocr}(D) \leq 2$. Let F be the set of i-odd edges in D. If it is true that
>
> (i) every cycle C in $G - F$ is crossing-free in D, and
>
> (ii) for each vertex $v \in V(C)$, v is incident to at most one edge on the interior of C and v is incident to at most one edge on the exterior of C,
>
> then G has a stable redrawing with at most $\mathrm{iocr}(D)$ crossings, in which $G - F$ is crossing-free.

The theorem follows from the main claim by applying it with $G = G'$, $D = D'$, $F = F'$. Figure 3 shows a drawing of a graph that fulfills the conditions of the main claim.

We establish the main claim by induction on the number of vertices plus the number of edges of G.

Let Y be the set of cut-edges of $G - F$. Then $G - (F \cup Y)$ consists of the union of cycles in $G - F$, together with any (isolated) vertices that are incident only to edges in $F \cup Y$. Thus $G - (F \cup Y)$ minus its isolated vertices is the edge-disjoint union of the 2-connected blocks (maximal 2-connected components) of $G - F$. Each face of $G - (F \cup Y)$ is bounded by a disjoint union of cycles and isolated vertices. If Y is drawn (or redrawn) without crossings, then the faces of $G - F$ correspond to the faces of $G - (F \cup Y)$, as described in the proof of Lemma 2.1.

Claim 1. We may assume that G is 2-connected, that the interior of each crossing-free cycle in D is empty in D, and that the outer face of $G - (F \cup Y)$ contains $F \cup Y$.

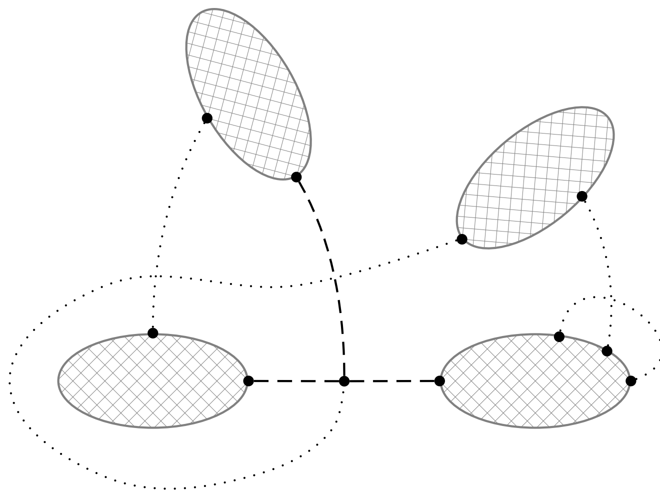


Figure 3: A graph G drawn such that F consists of two pairs of i-odd edges (dotted); cut-edges of $G - F$ are dashed, 2-connected components cross-hatched. Claim 1 establishes that we can assume that the interiors of the 2-connected components are empty and the outer face of $G-(F \cup Y)$ contains $F \cup Y$, where Y is the set of cut-edges of $G - F$.

Proof. If C is a crossing-free cycle with non-empty interior and exterior, then apply induction to D minus the interior and to D minus the exterior. The stable redrawings can be combined into a stable redrawing of G. Thus we may assume that C has either empty interior or empty exterior.

By considering the drawing on a sphere instead of a plane, we can consider any face of $G - (F \cup Y)$ to be the outer face of the drawing: let it be a face that intersects $F \cup Y$. (This may change the meaning of interior and exterior for a cycle C, but otherwise the redrawing is stable, and at the end of the proof, the original meanings can be reobtained via the sphere again.) The exterior of every cycle in $G - (F \cup Y)$ is non-empty, so every cycle in $G - (F \cup Y)$ has empty interior. Hence every face of $G - (F \cup Y)$, aside from the outer face, is the interior of a single cycle (as opposed to having more than one cycle in the face boundary). Therefore, the outer face of $G - (F \cup Y)$ contains all of $F \cup Y$.

If G is not connected, let H be a component. H lies on the exterior of each crossing-free cycle in $G - H$ (i.e., the exterior of each cycle in $G - H - (F \cup Y)$). Obtain stable redrawings of H and $G - H$ by induction. In the redrawing of $G - H$, there must be an open disk in the outer face of $G - H - (F \cup Y)$ that does not intersect $G - H$; insert the redrawing of H into that disk to obtain a stable redrawing of G.

If G is connected but not 2-connected, let H be a leaf-block with cut-vertex x. Redraw H and $G - (V(H) - x)$ by induction, and insert the first drawing into the exterior face of the second drawing restricted to $(G - (V(H) - x)) - F$, with no additional crossings, so that both copies of x are drawn at the same point. ☐

Claim 2. We may assume that F contains exactly two disjoint i-odd pairs in D; in particular, $|F| = 4$ and $\mathrm{iocr}(D) = 2$.

Proof. If F contains two disjoint i-odd pairs in D, then $|F| \geq 4$ and $\mathrm{iocr}(D) \geq 2$; since $|F| \leq 2\,\mathrm{iocr}(G) = 4$ and $\mathrm{iocr}(D) \leq 2$, we have $|F| = 4$ and $\mathrm{iocr}(D) = 2$ in this case.

If F does not contain exactly two disjoint i-odd pairs in D, then there is some edge $e \in F$ that is part of every edge pair contributing to $\mathrm{iocr}(D)$. Let $F' = F - e$ be the set of (one or two) edges that form i-odd pairs with e.

Since no two non-adjacent edges in $G - e$ cross oddly, we can apply Lemma 2.1, which produces a stable redrawing D^* of $G - e$ that is crossing-free. By Claim 1, the endpoints of e lie on the boundary of the outer face of $G - (F \cup Y)$, and a stable redrawing will not change that. The boundary of the outer face of $G - F$ in D^* consists of the edges and vertices in the boundary of the outer face of $G - (F \cup Y)$ plus the edges of Y, so the endpoints of e belong to its boundary, too. Adding the edges of F' to $D^* - F'$ one-by-one according to D^*, each edge divides at most one face into two. Then e can be drawn so that it does not cross $D^* - F'$ and it crosses each edge of F' at most once. Since $|F'| = \mathrm{iocr}(D')$, this suffices. ☐

Redrawings can result in edges with self-intersections; these can be easily removed by modifying the drawing of the edge locally near the self-intersection (see [9], for example).

Claim 3. We may assume that $G - F$ has no isolated vertices, each leaf of $G - F$ is incident to exactly two edges of F, and G has minimum degree at least 3.

Proof. If G has a vertex of degree one, then contract it to a neighbor and apply induction. We can then add the contracted edge to the drawing without creating any crossings.

Suppose that G has a vertex v of degree two, with incident edges e and f. If v is incident to an edge of F, let $f \in F$. Contract v along e to the other endpoint u of e, and remove any self-intersections. Any newly created (independent) odd pair would involve f and an edge f' that formed an (independent) odd pair with e prior to the contraction (so $f' \in F$), so F still contains all i-odd pairs. Apply induction to redraw. Reinsert v on f close to u, letting e be uv. This yields a stable redrawing of G in which $G - F$ is crossing-free. Thus, we may assume that G has minimum degree at least 3.

A vertex of G is incident to at most one edge from each i-odd pair, so it is incident to at most two edges of F. Therefore, a leaf of $G - F$ must be incident to exactly two edges of F, and there can be no isolated vertices of $G - F$. □

To complete the argument, we find it useful to extend the definition of rotation from single vertices to crossing-free connected subgraphs. Suppose that H is a component of $G - F$, and let D^* be a drawing of G such that H is crossing-free. The outer face boundary of H in $D^* - F$ is a closed facial walk W, oriented clockwise. As we traverse the outer face alongside W, we pass the ends of edges (or "half-edges") of F that are incident to H; let this cyclic ordering of ends of edges be called *the rotation at H in D^**.

The rotation at H is determined by the full drawing of G, but we also need a way to talk about potential rotations at H just based on the incidence of edges in F with H. So consider any crossing-free drawing of H, and let W be its outer face boundary. (We do not draw any edges not belonging to H.) Note that cut-vertices of H appear more than once in W. If v is an endpoint of $e \in F$ in H, and v appears multiple times in W, then let the end of e at v be assigned to any one of the copies of v in W. Repeat this for all ends of edges in F that are incident to H. If more than one edge end is assigned to the same element of W, then order them arbitrarily. This yields a cyclic ordering of the ends of edges in F incident to H, which we call *an abstract rotation at H*.

A rotation can be represented by a cyclic permutation of edges in F, with each edge appearing at most twice. If we wish to distinguish the two ends of an edge e, we write them as e_1 and e_2.

Remark 4. For use in upcoming proofs, we briefly consider the cyclic permutations of edges of F in which no edge appears twice: A set of size one or two has only one cyclic permutation. There are two distinct cyclic permutations of three elements a, b, c: abc and its reverse, acb. Since there

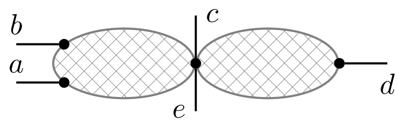


Figure 4: The rotation at H in the given drawing is the cyclic permutation (a, b, c, d, e); the abstract rotations at H are (a, b, c, e, d), (a, b, e, c, d), (a, b, c, d, e), (a, b, e, d, c), (a, b, d, c, e) and (a, b, d, e, c).

are $4!/4 = 6$ cyclic permutations of $\{a, b, c, d\}$, if we group each one with its reverse, then there are 3 distinct pairs: $\{abcd, adcb\}$, $\{abdc, acdb\}$, and $\{acbd, adbc\}$. Observe that we can switch between these pairs if we are allowed to swap consecutive elements: we can move from *abcd* or *adcb* to the second pair by swapping c and d, move from the second pair to the third pair by swapping b and d, and move from the third pair to the first pair by swapping b and c.

Claim 5. We may assume that for any component H of $G - F$, there exists an edge of F with both endpoints in H.

Proof. Suppose that H is a component of $G - F$ and no edge of F has both endpoints in H. Then $G - F$ has more than one component. By Claim 3, H has more than one vertex.

Let $F' \subseteq F$ be the set of i-odd edges incident to H. $|F'| \geq 2$ since G is 2-connected (by Claim 1).

Since H is connected, we may contract H within the plane to a vertex v_H by a sequence of $n(H) - 1$ edge contractions, deleting any loops created, and removing any self-intersections. Let G/H and D/H be the resulting (multi)graph and drawing, in which each edge of F' now has v_H as an endpoint. Two edges of F' can have the same endpoint in $G - H$, so G/H may have multiple edges. Since each pair of i-odd edges in F does not have shared endpoints, G/H can have up to two pairs of multiple edges, where each pair contains one edge from each pair of i-odd edges in F.

Any crossings created by the contraction are between two edges of F'. Since those edges all share the endpoint v_H in D/H, no new i-odd pairs are created, and any such pair that now shares the endpoint v_H is no longer i-odd. Therefore, $\text{iocr}(D/H) = 0$ if $|F'| = 4$, and $\text{iocr}(D/H) = 1$ if $|F'| = 3$. Remove one edge from each pair of multiple edges, apply induction to redraw, and then add back each removed edge, drawn in parallel to its multiple edge mate. Let D'/H be the resulting drawing of G/H.

If D/H has no multiple edges, then $\mathrm{cr}(D'/H) \leq \mathrm{iocr}(D'/H) \leq \mathrm{iocr}(D/H)$. If D/H has two pairs of multiple edges, then D'/H is crossing-free. Suppose then that D/H has one pair of multiple edges. Since every edge in F is part of an i-odd pair, removing one reduces iocr by 1, so immediately after applying induction there is at most one crossing, and therefore adding the edge back adds at most one crossing. So $\mathrm{cr}(D'/H) \leq \mathrm{iocr}(D/H)$ in this case, and in every case. Also, D'/H is a stable redrawing of D/H since F and v_H are in the outer face of $G - (F \cup Y)$ restricted to $G - H$.

Let $D''[H]$ be a stable redrawing of D restricted to H obtained by applying Lemma 2.1 to the drawing of H in D. Since H contains no edges of F, $D''[H]$ is crossing-free, and each endpoint of F' in H is on the boundary of its outer face. Fix an abstract rotation at H in $D''[H]$. If it is the same as the rotation at v_H in D'/H, then we can replace v_H in that drawing by $D''[H]$, obtaining a drawing of G whose only crossings are the crossings in D'/H. This completes the proof. If the rotation at H in $D''[H]$ is equal to the rotation at v_H reversed, then we can flip the drawing of H in the plane, which reverses its rotation so that it equals the rotation at v_H. Then we finish the proof as before. Otherwise, by Remark 4, v_H must be incident to all four edges in F (i.e., $|F'| = 4$). Recall that in this case D'/H was crossing-free. Also, as observed in the remark, by swapping two consecutive elements in the rotation at v_H, and possibly reversing the rotation, we can obtain the rotation at H. We make the swap by redrawing near v_H, which adds one crossing, and, if necessary, flip the drawing of H to reverse the rotation at H. Then we can replace v_H in D'/H by $D''[H]$ (possibly flipped) to get a drawing of G with exactly one crossing, and so that $G - F$ is crossing-free. In each case, we obtain a stable redrawing since flipping H will not switch the contents of the interior (exterior) of any cycle in $G - F$. □

Claim 6. We may assume that $G - F$ is connected.

Proof. By Claim 5, we can assume that for every component H of $G - F$, there is some edge $e \in F$ with both endpoints in H. Suppose that H' is another component of $G - F$, and let $e' \in F$ have both endpoints in H'. Let $\{f, f'\} = F - \{e, e'\}$. G is 2-connected by Claim 1, so f and f' must each connect H to H', and $G - F$ has no other components. $G - F$ has a stable redrawing with no crossings by Lemma 2.1, and the boundary of its outer face (which has two components, the boundary of the outer face of H and the boundary of the outer face of H') contains all endpoints of edges of F. Fix abstract rotations at H and at H'. Then f and f' can be added to the outer face (dividing the outer face into two faces) so that $G - \{e, e'\}$ is drawn crossing-free. If the ends f and f' alternate with the ends of e in the rotation at H, then e can be drawn near H so that it crosses only f, once; otherwise e can be drawn near H with no crossings. Likewise for e' near H', so we produce a drawing of G as desired. □

Let T be the block-cutpoint tree of $G - F$: In one partite set of T, each vertex is a block of $G - F$ (a maximal 2-connected subgraph of $G - F$), in the other partite set, each vertex is a cut-vertex of $G - F$. Adjacency in T is containment in $G - F$. Since $G - F$ cannot be a single vertex by Claim 3, each block is either a maximal 2-connected subgraph of $G - F$, or a cut-edge of $G - F$ with its endpoints. By the inductive assumption, each 2-connected block is bounded by a crossing-free cycle in D.

Each leaf of T is a block in $G - F$, called a *leaf-block* of $G - F$.

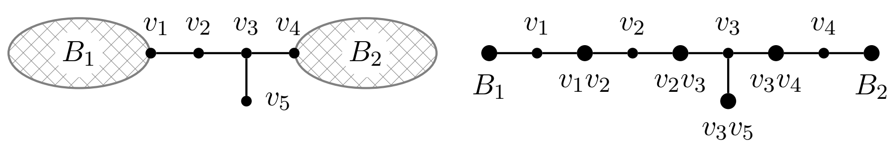


Figure 5: A graph with two 2-connected blocks, and its block-cutpoint tree.

Claim 7. We may assume that $G - F$ has more than one block; equivalently, $|V(T)| > 1$. Also, if H is a leaf-block and v is the cut-vertex of $G - F$ in H, then no edge of F has both endpoints in $H - v$.

Proof. We first show the claim about leaf-blocks. If H is a leaf-block with only two vertices (a cut-edge with its endpoints), then an edge of F with both endpoints in $H - v$ would be a loop, which is not possible, since G is simple.

Suppose then that H is a 2-connected leaf-block. By Claim 1 and condition (ii) of the main claim, no two edges of F are incident to the same vertex of $H - v$. Suppose that $e \in F$ has endpoints u, u' in $H - v$, and let P be the u, u'-path on the boundary of H that does not contain v. Choose e so that P is minimal; then there is no edge of F with both endpoints in P.

Suppose that there is an edge $f \in F$ with exactly one endpoint in P; this endpoint, x, is in $P - \{u, u'\}$. Since $G - F$ is connected, f extends via its other endpoint, through $G - F$, to an x, v-path Q. Since v, x alternates with u, u' along the cycle bounding H, Q crosses e oddly. Then Q contains an edge that crosses e oddly, and since Q does not contain u or u', this forms an i-odd pair; since Q contains only one edge of F, namely f, this edge has to be f. By Claim 2, there is at most one such edge.

By Lemma 2.1 or induction, $G-e$ has a stable redrawing with at most one crossing (between two other edges of F). Since H is 2-connected it is bounded by a cycle, so in the redrawing, the only edge incident to $P-\{u,u'\}$ in the exterior of H is f (if such an edge exists). Therefore, e can be added to the drawing so that e crosses no edge other than f, which it crosses once. Hence, in this case, we have found a stable redrawing of G with at most two crossings and we are done.

Finally, we have to show that we can assume that $G-F$ does not consist of a single block. $G-F$ cannot be a two-vertex block, since then every edge of F would be a multiple edge, which is impossible, since G is simple. Hence, if $G-F$ is a single block, it must be a 2-connected component and every edge of F has both endpoints on $G-F$. We can then apply the leaf-block argument replacing v with an arbitrary vertex on the boundary of H; in this case Q is not needed to show that e and f cross oddly. □

Claim 8. If H is a 2-connected leaf-block of $G-F$ and v is the cut-vertex of $G-F$ in H, then we may assume that $H-v$ is incident to all four edges of F.

Proof. Let F' be the set of edges in F that are incident to $H-v$. Assume that $|F'| \leq 3$.

By Claim 7, no edge of F has both endpoints in $H-v$. H is bounded by a cycle, so by Claim 1 and condition (ii) of the main claim, no two edges of F can be incident to the same vertex in H, and v is incident to exactly one edge vu of $G-F$ on the exterior of H (for some vertex u). Let R be the clockwise cyclic ordering of all edge ends at H in D. (The rotation at H in D only includes edges of F, so R is the usual rotation at H plus the edge vu.)

Let G' and D' be obtained from G and D by contracting $H-v$ within the plane to a vertex v_H. Then v_H is a leaf in $G'-F$. Temporarily ignoring all but one copy of each multiple edge in G', apply induction to get a stable redrawing; then multiple edges can be drawn near their remaining copies so that we obtain a stable redrawing D'' of G'. By essentially the same argument as in the proof of Claim 5, we have $\text{iocr}(D'') \leq \text{iocr}(D')$.

Consider the rotation at v_H in G', but with v_Hv replaced by vu: if this is the same as R or its reverse, then we can put H back in the drawing, possibly flipped, without adding crossings. Otherwise, by Remark 4, we must have $|R| > 3$, which implies that $|F'| = 3$. Thus we may assume that

R is vu, e_1, e_2, e_3. Since $|F| = 4$, two of the edges in $\{e_1, e_2, e_3\}$ form an odd pair, which is not independent after $H - v$ is contracted, so $\text{iocr}(D') < 2$, and hence D'' has less than two crossings. If the clockwise order around v_H in D'' is $v_H v, e_1, e_2, e_3$ or $v_H v, e_3, e_2, e_1$, we can insert H without adding crossings. Thus, by symmetry we may assume that the clockwise order around v_H in D'' is $v_H v, e_2, e_1, e_3$ or $v_H v, e_2, e_3, e_1$. In the former case we can replace vv_H by H, adding only one crossing between e_1 and e_2; in the latter case we insert H with orientation reversed (so v, e_3, e_2, e_1 appear in that clockwise order around H) and add one crossing between e_2 and e_3. Since $\text{iocr}(D'') < 2$, the total number of crossings is at most 2 and all those crossings are between edges of F. □

Claim 9. If every isolated vertex of $G - (F \cup Y)$ has degree at most 3 in G, then we may assume that $G - F$ is drawn with no crossings.

Proof. If a cut-edge e of $G - F$ crosses another edge f oddly, then e and f share an endpoint v. Each vertex of a cycle is incident to at most one edge with crossings, so v is not in any cycle of $G - F$. Then by assumption, v has degree at most 3 in G. The edges incident to v can be made to cross pairwise evenly by redrawing them near v (which may change the rotation at v). Therefore, if we repeat this for all such vertices v, all cut-edges in $G - F$ are now even. Since $G - (F \cup Y)$ remains crossing-free, with F still on the outer face of $G - (F \cup Y)$, applying Lemma 2.2 yields a stable redrawing of G such that $G - F$ is crossing-free, with no new i-odd pairs. □

Let $F = \{a, b, c, d\}$. We use $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ to label the ends of edges in F.

Claim 10. We may assume that T is a path.

Proof. If T is not a path, there are at least 3 leaf-blocks. Claims 8 and 3 imply that every 2-connected leaf-block H with cut-vertex v is incident to four ends of F, and every other leaf-block has a leaf of G that is incident to two ends of F. Since F has 8 edge ends, $G - F$ has either one 2-connected leaf-block and two leafs, or $G - F$ has no 2-connected leaf-blocks and three or four leafs.

For any stable redrawing of $G - F$, the endpoints of F will still be on the boundary of the outer face. Suppose that $G - F$ has a stable redrawing which has an abstract rotation for which the ends of some edge $a \in F$ are consecutive in that rotation. Then the edges of F can be drawn such that a is crossing-free and every two edges in $\{b, c, d\}$ cross at most once. This gives less than three crossings unless the rotation is $a_1 a_2 b_1 c_1 d_1 b_2 c_2 d_2$ (without loss of generality). That rotation can be avoided if there exists a leaf of $G-F$ that is not incident to a, since swapping the ends at that leaf gives another rotation pattern. Otherwise, a_2 and b_1 are at one leaf of $G-F$, d_2 and a_1 are at another leaf of $G-F$, and the other ends are at $H-v$, where H is a 2-connected block of $G-F$ with cut-vertex v. But then we can flip the drawing of H to get a new drawing of $G-F$ with rotation $a_1a_2b_1c_2b_2d_1c_1d_2$. Thus we may assume that no edge has consecutive ends in any abstract rotation of any stable redrawing of $G-F$ (including the rotation in D).

Consider the case that $G-F$ has four leafs. Each leaf is incident to two edges of F. By the previous paragraph we may assume the leafs of $G-F$ are incident to edge pairs $\{a,b\}$, $\{c,d\}$, $\{a,b\}$, and $\{c,d\}$, in that order along the outer face of $G-F$. Also, there is no good redrawing of $G-F$ that would swap the positions of leafs incident to different edge pairs, so $G-F$ cannot have a cut-vertex that separates the leafs.

We wish to apply Claim 9, so consider any isolated vertex v of $G-(F\cup Y)$. If v is a leaf of $G-F$ then it has degree 3 in G. Otherwise, v is not incident to any edges of F, only edges in Y. Such an edge leads to a leaf-block of $G-F$, which contains a leaf of $G-F$; since v cannot separate the four leafs, v must have degree less than 4. Thus we may apply Claim 9, so $G-F$ is crossing-free in D. However, the rotation implies that each edge a,b must cross each edge c,d oddly, giving four i-odd pairs in D, a contradiction.

Thus, we may assume that $G-F$ does not have four leafs. So $G-F$ has exactly 3 leaf-blocks.

Suppose that $G-F$ has a 2-connected leaf-block H with cut-vertex v. Then by Claims 8 and 3, $G-F$ has two leafs, and each edge of F has one end at $H-v$ and the other end at a leaf. Then Claim 9 applies, so $D-F$ is crossing-free. If the rotation at $H-v$ is $abcd$, then to avoid consecutive ends, the rotation must be $abcd(ab)(cd)$, where elements within parentheses can be swapped because they are incident to the same leaf of $G-F$. But then $\{a,c\}$, $\{a,d\}$, and $\{b,d\}$ are all i-odd pairs in D, a contradiction.

This means that $G-F$ has 3 leafs and no 2-connected leaf-blocks. Six ends of edges in F are at the leafs, and two ends are not. There must be two edges a,b that have both ends at the leafs. Since a and b do not have consecutive ends in any abstract rotation, the rotation at $G-F$ is $(xa_1)y(a_2b_1)z(b_2w)$, where elements in parentheses are incident to the same leaf of $G-F$ (and can be swapped), and $\{x,y,z,w\}=\{c_1,c_2,d_1,d_2\}$. Also, c and d may not have consecutive ends in any abstract rotation at $D-F$, so the rotation is $(c_1a_1)d_1(a_2b_1)c_2(b_2d_2)$ without loss of generality. If d_1 or c_2 is at a cut-vertex of $G-F$, then this drawing has another abstract rotation which differs just in the position of d_1 or c_2; this, however, is a rotation pattern we already covered in an earlier case. Hence we may assume that no cut-vertex of $G - F$ is incident to any ends of edges in F. Since T has maximum degree 3, any vertex of $G - F$ not in a cycle has degree at most 3. The previous two sentences mean that Claim 9 applies, so $G - F$ was not actually redrawn. But then $\{a, d\}$, $\{c, d\}$, and $\{b, c\}$ are all i-odd pairs in D, a contradiction. □

Claim 11. We may assume that $G - F$ is crossing-free in D.

Proof. If a cut-edge e of $G - F$ crosses another edge f oddly, then e and f share an endpoint v. Each vertex of a cycle is incident to at most one edge with crossings, so v is not in any cycle of $G - F$. Then by Claim 10, v is incident to exactly two edges of $G - F$, say e and f. We can redraw edges near v so that e and f cross all edges incident to v evenly, using the method in Lemma 2.3 (Figures 1 and 2). These redrawing moves do not create any i-odd pairs. Repeat for every vertex v that is not in any cycle of $G - F$. Since every cut-edge of $G - F$ was i-even, they are now even, and $G - F$ is now even. Now apply Lemma 2.2 to get a stable redrawing of G such that $G - F$ crossing-free, with no new i-odd pairs. □

Claim 12. We may assume that F has no stable redrawing with at most two odd pairs such that $G - F$ remains crossing-free.

Proof. Consider the rotation at $G - F$ for such a drawing. Two edges of F form an odd pair if and only if their ends alternate in the rotation, since $G - F$ is connected.

It is easy to redraw F with the same rotation in the outer face of $G - F$ such that each pair of edges crosses at most once. Since two edges of F that cross at most once, cross if and only if their ends alternate in the rotation at $G - F$, we have obtained a stable redrawing of G with at most two crossings. □

We can now complete the proof of Theorem 3.1.

By Claim 7 and 10, $G - F$ has exactly two leaf-blocks. If both are 2-connected, then by Claim 8 the edges of F have eight distinct endpoints in $G - F$. Then each odd pair is actually an i-odd pair, which contradicts Claim 12. Suppose that H is a 2-connected leaf-block of $G - F$ and v is the cut-vertex of $G - F$ in H. We may assume that the rotation at $G - F$ restricted to H is $a_1 b_1 c_1 d_1$. $G - F$ has a leaf v' incident to distinct edges $x, y \in F$, and let $\{z, w\} = F - \{x, y\}$ (with $\{x, y, z, w\} = \{a, b, c, d\}$). We can redraw near v' so that x, y is an even pair, without creating i-odd pairs.

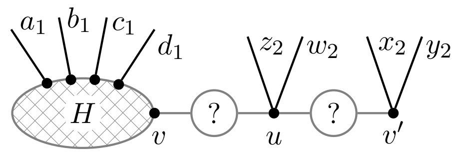


Figure 6: $G - F$, shown with a possible rotation.

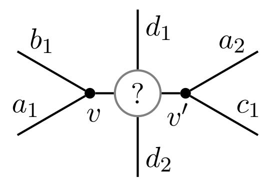


Figure 7: $G - F$ with all ends except b_2 and c_2.

If z_2 and w_2 (the ends of z and w that are not in $H - v$) are at distinct vertices, then there are exactly two odd pairs, contradicting Claim 12. Thus we may let u be the vertex shared by z_2 and w_2. Then u is not in any cycle of $G-F$, so the drawing has a rotation $a_1b_1c_1d_1((x_2y_2)z_2w_2)$, where the last four ends can be reordered arbitrarily as long as x_2 and y_2 are consecutive (Figure 6). No matter how a, b, c, d maps to x, y, z, w, we may draw x and y without adding crossings: If $\{x, y\}$ is $\{a, b\}$, $\{b, c\}$, or $\{c, d\}$, we can add z and w to the drawing without crossings. If $\{x, y\}$ is $\{a, c\}$ or $\{b, d\}$, we draw z and w with one crossing, and if $\{x, y\}$ is $\{a, d\}$, then z and w can be drawn with one crossing each, such that $G - F$ remains crossing-free.

Now we can assume that $G-F$ has 2 leafs v, v', and no 2-connected leaf-blocks. Suppose that a is incident to both leafs; then we may assume that a_1, b_1 are incident to v and a_2, c_1 are incident to v'. Then $\{a, d\}$ must be an i-odd pair, so the rotation at $G - F$ (Figure 7) contains $(a_1b_1)d_1(a_2c_1)d_2$ as a cyclic subsequence, where elements in each pair of parentheses might be in reverse order. Since $\{b, c\}$ is the other i-odd pair, the rotation at $G-F$ is obtained from the above cyclic sequence by replacing d_1 by $c_2b_2d_1$, $c_2d_1b_2$, or $d_1c_2b_2$, or by replacing d_2 by $b_2c_2d_2$, $b_2d_2c_2$, or $d_2b_2c_2$. In each case, either b or c will form an i-odd pair with d, a contradiction.

Thus we may assume that a_1 and b_1 are incident to v, and that c_1 and d_1 are incident to v'. If the rotation at $G-F$ has the form $(a_1b_1)x_2y_2(c_1d_1)z_2w_2$, then there is a drawing with no crossings if $\{x, y\}$ equals $\{a, b\}$ or $\{c, d\}$, and otherwise there is a drawing where there are no crossing pairs of edges except x, y and z, w (which may or may not be crossing pairs). If the rotation at $G - F$ is $(a_1b_1)x_2y_2z_2(c_1d_1)w_2$, then there is an abstract rotation where the w_1 and w_2 are consecutive; assuming without loss of generality that $w = d$, $F - c$ can be drawn without crossings, and then c can be drawn with at most two crossings.

So we may assume that the rotation has the form $(a_1b_1)x_2y_2z_2w_2(c_1d_1)$. By flipping the drawing of blocks of $G - F$ as needed, one can obtain a drawing of $G - F$ with a rotation of a different pattern (which was already ruled out) *unless* there is a 2-connected block H with cut-vertices u, u' such that x_2, y_2, z_2, and w_2 are incident to (distinct) vertices of (one component of) $H - \{u, u'\}$. By redrawing the edge ends near v and v', we can make a, b and c, d even pairs. This contradicts Claim 12. □

4 An Open Problem

While we now know that the independent odd crossing number is polynomially bounded within the crossing number of a graph, we do not know if every graph has a drawing realizing the independent odd crossing number which has a polynomial number of crossings. Indeed, it is not even clear whether there is any bound on the number of crossings in an iocr-optimal drawing that depends on the independent odd crossing number only, and not on the size of the graph. For the odd crossing number we were able to show such a result: every graph G has an ocr-optimal drawing with at most $9^{\mathrm{ocr}(G)}$ many crossings [11]. We used this result to show that ocr is fixed-parameter tractable (extending work of Grohe for crossing numbers).

References

[1] Peter Brass, William Moser, and János Pach. *Research Problems in Discrete Geometry*. Springer, New York, 2005.

[2] Serguei Norine. Pfaffian graphs, T-joins and crossing numbers. *Combinatorica*, 28(1):89–98, 2008.

[3] János Pach and Micha Sharir. *Combinatorial geometry and its algorithmic applications: The Alcalá lectures*, volume 152 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009.

[4] János Pach and Géza Tóth. Thirteen problems on crossing numbers. *Geombinatorics*, 9(4):194–207, 2000.

[5] János Pach and Géza Tóth. Which crossing number is it anyway? *J. Combin. Theory Ser. B*, 80(2):225–246, 2000.

[6] János Pach and Géza Tóth. Degenerate crossing numbers. In *Computational geometry (SCG'06)*, pages 255–258. ACM, New York, 2006.

[7] Michael Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Crossing numbers of graphs with rotation systems. *Algorithmica*, 2009.

[8] Michael J. Pelsmajer, Marcus Schaefer, and Despina Stasi. Strong Hanani–Tutte on the projective plane. *SIAM Journal on Discrete Mathematics*, 23(3):1317–1323, 2009.

[9] Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Removing even crossings. *J. Combin. Theory Ser. B*, 97(4):489–500, 2007.

[10] Michael J. Pelsmajer, Marcus Schaefer, and Daniel Stefankovic. Removing even crossings on surfaces. *European Journal of Combinatorics*, 30(7):1704 – 1717, 2009.

[11] Michael J. Pelsmajer, Marcus Schaefer, and Štefankovič. Crossing numbers and parameterized complexity. In *Graph Drawing*, pages 31–36, 2007.

[12] Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Odd crossing number and crossing number are not the same. *Discrete Comput. Geom.*, 39(1):442–454, 2008.

[13] László A. Székely. A successful concept for measuring non-planarity of graphs: the crossing number. *Discrete Math.*, 276(1-3):331–352, 2004.

[14] László A. Székely. An optimality criterion for the crossing number. *Ars Math. Contemp.*, 1(1):32–37, 2008.

[15] Terence Tao and Van Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[16] Géza Tóth. Note on the pair-crossing number and the odd-crossing number. *Discrete Comput. Geom.*, 39(4):791–799, 2008.

[17] William T. Tutte. Toward a theory of crossing numbers. *J. Combinatorial Theory*, 8:45–53, 1970.

[18] Pavel Valtr. On the pair-crossing number. In *Combinatorial and Computational Geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 569–575. Cambridge University Press, Cambridge, 2005.

[19] Hein van der Holst. Algebraic characterizations of outerplanar and planar graphs. *European J. Combin.*, 28(8):2156–2166, 2007.