The Degenerate Crossing Number and Higher-Genus Embeddings

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Abstract. If a graph embeds in a surface with k crosscaps, does it always have an embedding in the same surface in which every edge passes through each crosscap at most once? This well-known open problem can be restated using crossing numbers: the degenerate crossing number, $\mathrm{dcr}(G)$, of G equals the smallest number k so that G has an embedding in a surface with k crosscaps in which every edge passes through each crosscap at most once. The genus crossing number, $\mathrm{gcr}(G)$, of G equals the smallest number k so that G has an embedding in a surface with k crosscaps. The question then becomes whether $\mathrm{dcr}(G) = \mathrm{gcr}(G)$, and it is in this form that it was first asked by Mohar.

We show that $\mathrm{dcr}(G) \leq 6\,\mathrm{gcr}(G)$, and $\mathrm{dcr}(G) = \mathrm{gcr}(G)$ as long as $\mathrm{dcr}(G) \leq 3$. We can separate dcr and gcr for (single-vertex) graphs with embedding schemes, but it is not clear whether the separating example can be extended into separations on simple graphs. Finally, we show that if a graph can be embedded in a surface with crosscaps, then it has an embedding in that surface in which every edge passes through each crosscap at most twice. This implies that dcr is **NP**-complete.

Keywords: degenerate crossing number, non-orientable genus, genus crossing number.

1 Introduction

When defining the crossing number of a graph, one typically requires that at most two edges cross in any point. If $k > 2$ edges cross in a single point, these edges can be perturbed slightly to create $\binom{k}{2}$ crossings of pairs of edges, so multiple crossings in a single point can always be avoided. Günter Rote and M. Sharir, according to Pach and Tóth [10] asked "what happens if multiple crossings are counted only *once*". This led Pach and Tóth to introduce the degenerate crossing number: we allow drawings which are *degenerate* in the sense that more than two edges are allowed to cross in a single point (but which are otherwise standard, in particular, edges have to actually cross, not touch, and self-crossings are not allowed). The *degenerate crossing number* of the drawing is the number of crossing points in the drawing. The *degenerate crossing number*, $\mathrm{dcr}(G)$, of a graph G is the smallest degenerate crossing number of any (degenerate) drawing of G. Some papers (e.g. [1]) restrict drawings to be simple, that is, every two edges intersect (or cross, that's not always clearly defined[3]) at most once; to distinguish this variant from dcr we call it the *simple degenerate crossing number*, $\text{dcr}^*(G)$.[4]

If we modify the definition of the degenerate crossing number to allow self-crossings of edges, we obtain the *genus crossing number*, $\text{gcr}(G)$, which was introduced by Mohar [8]. By definition, $\text{gcr}(G) \leq \text{dcr}(G)$. Mohar conjectured that $\text{gcr}(G) = \text{dcr}(G)$. Equality of these two numbers would be particularly interesting, since, as Mohar observes, $\text{gcr}(G) = \widetilde{\gamma}(G)$, where $\widetilde{\gamma}(G)$ is the *non-orientable genus* (or the *minimum crosscap number*) of G, the smallest number k so that G can be embedded on a surface with k crosscaps (we allow the special case of $k = 0$ for planar graphs). Each crossing of multiple edges can be replaced by a crosscap and vice versa, since edges have to cross (and may not touch) in a crossing point. Similarly, $\text{dcr}(G)$ can be viewed (as we did in the abstract) as the smallest number k so that G has an embedding on a surface with k crosscaps so that every edge passes through each crosscap at most once. An edge not being allowed to pass through a crosscap more than once corresponds to prohibiting self-crossings in degenerate drawings in the plane. We view crosscaps as geometric, rather than purely topological objects, a view which we believe makes sense in graph drawing, where we need to visualize objects.[5]

We do not yet know, whether $\text{gcr}(G) = \text{dcr}(G)$ in general, but we can separate them, if we are allowed to equip graphs with an embedding scheme (a fixed rotation at each vertex, and a signature for each edge). In that case, there are graphs for which gcr is 3, but dcr is 4 as we will see in Theorem 4.

Remark 1 (Visualizing Graphs in Higher-Order Surfaces). Whether gcr = dcr or not has consequences for visualizing graphs embeddable in higher-order surfaces in the plane. Typically, such graphs are visualized using a (canonical) polygonal schema. There are polynomial-time algorithms for this task, e.g., see [4] for orientable surfaces, also see [3, 7, 5]. Assuming that vertices may not lie on the boundary (of the polygonal schema), the question gcr = dcr then becomes: do edges have to pass through the same side of a schema more than once? Many of the visualization algorithms (including [3]) start by contracting the graph to a single-vertex graph with an embedding scheme; for these algorithms, the example in Theorem 4 shows that edges can be forced to cross through the same side more than once.

On the other hand, we can show that $\text{dcr}(G) \leq 6\,\text{gcr}(G)$, so any graph embeddable in a surface with k crosscaps can be embedded in a surface with at most $6k$ crosscaps so that every edge passes through each crosscap at most once. We will establish this in Theorem 2. If we allow an edge to pass through

[3] The difference is that a shared endpoint counts as an intersection, but not a crossing.
[4] An example in the entry on degenerate crossing number in [12] shows that it matters whether dcr* is defined so as to allow crossings between adjacent edges or not.
[5] Mohar [8] uses a "planarizing system of disjoint 1-sided curves to define "passing through a crosscap" formally.

each crosscap just twice, it turns out that every graph can then be embedded in a surface with $\widetilde{\gamma}(G)$ crosscaps (Theorem 5).

1.1 Known Results

Pach and Tóth [10] showed that $\mathrm{dcr}(G) < |E(G)|$. For the simple degenerate crossing number, a crossing lemma is known: $\mathrm{dcr}^*(G) \geq c \cdot |E(G)|^3/|V(G)|^2$ for $|E(G)| \geq 4|V(G)|$ (and some constant $c > 0$). This was shown by Ackerman and Pinchasei [1], improving an earlier result by Pach and Tóth. We should also mention work by Harborth [6], who may have been the first to study multiple crossings in drawings. His goal is to maximize the number of multiway crossings. For example, he shows that K_{2m} can be drawn with two m-fold crossings; he conjectured that K_{2m} cannot be drawn with three or more m-fold crossings.

2 Tools

We start with some basic facts about (simple) closed curves on a non-orientable surface S. A closed curve C is called *non-separating* if $S - C$ consists of a single piece. Otherwise, C is *separating*. If it is separating, it can be *contractible* (one of the two pieces is homeomorphic to a disk) or *surface-separating*. The *sidedness* of a closed curve is the number of sides it has: it is either *one-sided* (its neighborhood is a Moebius strip) or *two-sided*. A closed curve C in a non-orientable surface *maximal* if $S - C$ is orientable (equivalently, if C passes through every crosscap an odd number of times).[6]

A surface can contain only a small number of different types of closed curves. The following lemma makes this precise.

Lemma 1 (Malnič, Mohar [9, Proposition 4.2.7]). *If G is a graph embedded in a surface S, and $\mathcal{P}$ is a collection of internally disjoint paths between vertices a and b (where $a = b$ is allowed), so that no two of the paths bound a disk in S, then*

$$|\mathcal{P}| \leq \begin{cases} 3\,\widetilde{\gamma}(S) - 2 & \text{if } \widetilde{\gamma}(S) \geq 2 \\ \widetilde{\gamma}(S) + 1 & \text{otherwise.} \end{cases}$$

Remark 2. We are interested in the case where $a = b$ and there are no surface separating paths; a better upper bound for that case would improve the upper bound in Theorem 4.

We also need some tools from topological graph theory to describe and handle embeddings of graphs on non-orientable surfaces. On orientable surfaces, an embedding can be described by a *rotation system* which prescribes a *rotation* (a clockwise, cyclic ordering) of the ends of all edges incident to a vertex. On non-orientable surfaces, we also need to prescribe, for every edge, its *signature*,

[6] There seems to be no standard name for curves of this type in the literature. Bojan Mohar suggests "orienting".

which is a number in $\{-1, 1\}$. A cycle in G is *two-sided* if the signature of its edges multiply to 1, otherwise, it is *one-sided*. A rotation system ρ and signature λ together form an *embedding scheme* (ρ, λ) of a graph on a surface. A drawing of a graph in a surface *realizes* an embedding scheme (ρ, λ), if the rotation at each vertex is as prescribed by ρ, and the sidedness of each cycle is as prescribed by the signatures of the edges. The sidedness of a cycle is determined by the parity of how often the cycle passes through crosscaps. Typical operations on graphs (removing/adding a vertex/edge, contracting an edge) are easily performed on the embedding scheme as well. For details, see [9, Section 3.3].

Arguments and algorithms for graph embeddings can often be simplified by replacing an embedded graph with a single-vertex graph with embedding scheme. This is often done for visualizing embeddings of graphs in higher-genus surfaces in the plane (see Remark 1). Note that in a single-vertex graph every edge is a loop, hence a closed curve, and we can talk about the sidedness, which then directly correspond to its signature: a one-sided loop has signature -1, and a two-sided loop signature 1. For a graph G with embedding scheme (ρ, λ) we define $\operatorname{gcr}(G, \rho, \lambda)$ as the smallest number k so that G has an embedding realizing (ρ, λ) on a surface with k crosscaps. Similarly, $\operatorname{dcr}(G, \rho, \lambda)$ is the smallest degenerate crossing number of any drawing of G which realizes the embedding scheme (ρ, λ).

The next lemma shows that as far as gcr is concerned, we can replace a graph with a graph on a single vertex equipped with an embedding scheme. For dcr we can do so for upper bounds only.

Lemma 2. *For every graph G there is a single-vertex graph G' with embedding scheme (ρ, λ) so that* $\operatorname{gcr}(G) = \operatorname{gcr}(G', \rho, \lambda)$, *and* $\operatorname{dcr}(G) \leq \operatorname{dcr}(G', \rho, \lambda)$.

Proof. Fix an embedding of G on a surface S with $k = \operatorname{gcr}(G)$ crosscaps. We can assume that G is connected (if it is not, we can extend G to a triangulation of S). Let T be a spanning tree of G. Contract edges of T, merging rotations in the embedding scheme at vertices that are identified and updating signatures of edges. Let G' be the resulting single-vertex graph with embedding scheme (ρ, λ). Then $\operatorname{gcr}(G', \rho, \lambda) \leq \operatorname{gcr}(G)$. If $\operatorname{gcr}(G', \rho, \lambda) < \operatorname{gcr}(G)$ were true, we could undo the operations which turned G into G' (since we maintained the embedding scheme) to find an embedding of G on a surface with less than $\operatorname{gcr}(G)$ crosscaps, which is a contradiction, so $\operatorname{gcr}(G) = \operatorname{gcr}(G', \rho, \lambda)$. The same argument shows that $\operatorname{dcr}(G', \rho, \lambda) < \operatorname{dcr}(G)$ is not possible, so $\operatorname{dcr}(G) \leq \operatorname{dcr}(G', \rho, \lambda)$. ∎

Note that we do not claim that $\operatorname{dcr}(G) \geq \operatorname{dcr}(G', \rho, \lambda)$, the construction we used may force an edge through a crosscap multiple times, so dcr can increase. Lemma 2 allows us to replace a graph with a single-vertex graph when showing that dcr can be bounded in gcr.

Finally, we need some basic techniques to deal with curves in a surface.

Theorem 1 (Weak Hanani-Tutte Theorem for Surfaces [2, 11]). *If G is drawn in a surface so that every pair of edges crosses an even number of times, then G has an embedding on the same surface with the same embedding scheme.*

It is well-known that a handle and two crosscaps are equivalent in the presence of another crosscap. So a graph embeddable on a surface with h handles can be embedded on a surface with $2h+1$ crosscaps so that every edge is two-sided. The following lemma shows that the odd number of crosscaps is not accidental when restricting to orientable embeddings, where we call an embedding (G, ρ, λ) of a graph G *orientable* if all cycles in G are two-sided (equivalently, multiplying the signatures of edges along each cycle, one always gets 1). Note that if G is a single-vertex graph, then its embedding is orientable, if all loops have signature 1.

Lemma 3. *Suppose k is minimal so that a connected graph (G, ρ) with rotation ρ has an orientable embedding on a surface with k crosscaps. Then either $k = 0$, or $k \geq 3$ and k is odd.*

Proof. Fix an orientable embedding of (G, ρ) in a surface with k crosscaps, where k is minimal. We can assume that G is a single-vertex graph (contract edges of a spanning tree, this leaves the embedding orientable, so $\lambda(e) = 1$ for all loops e now). Suppose k is even. Let c be one of the crosscaps. For any edge that passes oddly through c, push that edge over all crosscaps. Note that pushing an edge over all crosscaps does not change the parity of crossing between any pair of edges since the number of crosscaps is even and every edge initially crosses through an even number of crosscaps oddly, and this remains true. At the end of this operation we have a drawing of G in which every pair of edges crosses an even number of times, and all edges pass through c an even number of times. We can then push all edges off c, again maintaining that every pair of edges crosses evenly. Now, by Theorem 1, (G, ρ, λ) has an orientable embedding in the surface with $k - 1$ crosscaps, so k cannot have been even if it was minimal. If $k = 2$, then an orientable embedding on the projective plane implies that the graph is planar (since every edge passes through the single crosscap an even number of times).

Corollary 1. *If a single-vertex graph (G, ρ) has an orientable embedding on a non-orientable surface with $k \geq 2$ crosscaps, we can add a one-sided loop into its embedding scheme, without changing the surface.*

Proof. Let $k' \leq k$ be minimal so that (G, ρ) has an orientable embedding on the surface with k' crosscaps. If $k' = 0$, then we can add two crosscaps, and a loop that passes through one of them; since $k \geq 2$ this is sufficient. Otherwise, by Lemma 3 we can assume that k' is odd and at least 3. To G add a loop with its ends consecutive in the rotation. Now push this loop once over each crosscap. Since all other loops are two-sided, every pair of edges crosses evenly, so by Theorem 1 the graph embeds in the surface with the same embedding scheme. The loop we added is one-sided and maximal.

3 Removing Self-Crossings

Theorem 2. $\mathrm{dcr}(H) \leq 6\,\mathrm{gcr}(H)$.

So a degenerate drawing with self-crossings can be cleaned of self-crossings at the expense of increasing the degenerate crossing number by a factor of six. We will make use of the following lemma.[7]

Lemma 4. $\mathrm{dcr}(H) \leq 2|E(H)|$.

Proof. Use Lemma 2 to create a single-vertex graph G on vertex v with embedding scheme (ρ, λ) so that $\mathrm{dcr}(H) \leq \mathrm{dcr}(G, \rho, \lambda)$. We proceed by induction on $|E(G)| = |E(H)|$. If $|E(G)| = 0$, there is nothing to show, so G has at least one loop. Pick a loop e whose ends at v are *closest* in the sense, that no other edge begins and ends in the wedge formed by the two ends of e (we direct e to differentiate between the two parts of the rotation system at v enclosed by the ends of v). If we can, we pick e one-sided. Suppose e is one-sided. Let (G', ρ', λ') be the graph obtained from G by reversing the order of the edges enclosed in the wedge formed by e (we "flip" the wedge), changing all their signatures (since every edge has at most one end in the wedge that flips the signature of every edge which has an end in the wedge), and removing e. By induction $\mathrm{dcr}(G', \rho', \lambda') \leq 2|E(G')|$. We can now add a crosscap close to v and pass all edges in the former wedge through that crosscap, reattaching them to v in their original order. This also reestablishes the original signatures of edges in G. Finally, we add back e in its proper place in the rotation, passing it through the crosscap once. By construction, $\mathrm{dcr}(G, \rho, \lambda) \leq 1 + \mathrm{dcr}(G', \rho', \lambda') \leq 2|E(G)|$.

If there is no closest, one-sided loop, e must be two-sided. Let $\tilde{G}$ be the same as G with one modification: let $\tilde{\lambda}(e) = -1$ and proceed as in the first case. We obtain a graph $\tilde{G}'$ so that $\mathrm{dcr}(\tilde{G}, \rho, \tilde{\lambda}) \leq 1 + \mathrm{dcr}(\tilde{G}', \rho', \tilde{\lambda}')$. Now add one additional crosscap passing only edge e through it, making it two-sided again. This shows that $\mathrm{dcr}(G, \rho, \lambda) \leq 1 + \mathrm{dcr}(\tilde{G}, \rho, \tilde{\lambda}) \leq 2 + \mathrm{dcr}(\tilde{G}', \rho', \tilde{\lambda}') \leq 2|E(G)|$. ■

Proof (of Theorem 2). Let H be a graph with $\mathrm{gcr}(H) = k$. Fix an embedding of H on a surface S with k crosscaps. By Lemma 2 we can transform H into a graph G on a single vertex v with an embedding scheme (ρ, λ) so that $\mathrm{gcr}(H) = \mathrm{gcr}(G, \rho, \lambda)$ and $\mathrm{dcr}(H) \leq \mathrm{dcr}(G, \rho, \lambda)$. We show the result by induction on $|E(G)| = |E(H)|$.

If $|E(G)| \leq 3k$, then the result follows from Lemma 4. So we can assume that $|E(G)| > 3k$. Lemma 1 implies that in this case there are two loops e and f so that $e \cup f$ bounds a disk (e and f are homotopic). Remove the disk (with any loops it may contain) from the surface, and identify e and f. Since this removes at least one edge from G we can apply induction to the resulting graph G'. From G' we can reconstruct an embedding of G by splitting e, f into two loops and reinserting the disk. Any loops in the disk which are not homotopic to e and f can be drawn close to v (so they do not use any crosscaps that e and f may be using). Any loops parallel to e and f use the same crosscaps as e and f, so in

[7] This approach was suggested by one of the reviewers, and simplifies the original proof.

the resulting drawing no edge uses any crosscap more than once (note that any such loops have the same signature as e and f, since e and f bound a disk). ■

Since the proof works with single-vertex graphs with embedding schemes, the separation of gcr and dcr for those types of graphs (Theorem 4) implies that the proof approach in Theorem 2 will not yield gcr = dcr, but we can prove equality for small values.

Theorem 3. *If* $\mathrm{dcr}(G) \leq 3$*, then* $\mathrm{gcr}(G) = \mathrm{dcr}(G)$.

For graphs with embedding scheme, this result is sharp, as Theorem 4 shows.

Proof. Since $\mathrm{gcr}(G) \leq \mathrm{dcr}(G)$ it is sufficient to show that if $\mathrm{gcr}(G) \leq 2$, then $\mathrm{dcr}(G) \leq \mathrm{gcr}(G)$. By Lemma 2 it is sufficient to prove the result for single-vertex graphs with embedding scheme: for G there is a single-vertex graph G' and an embedding scheme (ρ, λ) so that $\mathrm{dcr}(G) \leq \mathrm{dcr}(G', \rho, \lambda)$ and $\mathrm{gcr}(G', \rho, \lambda) = \mathrm{gcr}(G)$, so establishing $\mathrm{dcr}(G', \rho, \lambda) \leq \mathrm{gcr}(G', \rho, \lambda)$ will prove the result.

If $\mathrm{gcr}(G', \rho, \lambda) = 0$, there is nothing to prove. If $\mathrm{gcr}(G', \rho, \lambda) = 1$ all loops are either two-sided and contractible or one-sided. Pick a closest loop e (in the sense defined in Lemma 4: every edge has at most one end in the wedge formed by e). If e is one-sided, we can proceed as in Lemma 4, cutting along e, flipping the wedge enclosed by e and changing the signature of all edges in the wedge. The resulting graph is embedded in a plane, and we can add back e so that it, and the edges it encloses cross through the crosscap exactly once. If e is two-sided, the ends of e must be consecutive. We can then remove e from the drawing, inductively draw the remaining graph, and add e back locally without using any crosscaps. If $\mathrm{gcr}(G', \rho, \lambda) = 2$, there may be two-sided loops which are not contractible. However, if there is a closest one-sided loop, or a closest two-sided loop which is contractible, we can proceed as in the case of a single crosscap. Hence, all closest loops are two-sided, and either separating, or maximal. Suppose there is a one-sided loop f. Then the wedge enclosed by f must contain both ends of another loop e. Pick e so it is closest (within the wedge formed by f). Now e cannot be maximal, since the ends of a maximal loop alternate with the ends of a one-sided loop in the rotation. Hence e is separating. But then anything starting inside the wedge formed by e must end within the wedge as well, so since e was chosen to be closest, its ends have to be consecutive in the rotation. We can then remove e, inductively draw the remaining graph, and add e back into the rotation without using any additional crosscaps. We conclude that there is no one-sided loop f, so all loops are two-sided. By Lemma 3, the graph is planar in this case. ■

A closer look at the proof of Theorem 2 and Theorem 3 show that they are purely combinatorial, and the bounds can be implemented algorithmically.

4 Separating dcr and gcr with Embedding Schemes

Theorem 4. *There is a single-vertex graph G with embedding scheme (ρ, λ) for which* $3 = \mathrm{gcr}(G, \rho, \lambda) < \mathrm{dcr}(G, \rho, \lambda) = 4$.

Proof. See the graph pictured in Figure 1(a). The single vertex is drawn as the outer cycle, to make the picture easier to read. So there are 5 loop edges $e_1, \ldots, e_5$ in this graph, the rotation at v is $e_1, e_2, e_3, e_4, e_5, e_3, e_2, e_1, e_4, e_5$, and the signatures are as in the embedding: $\lambda(e_1) = \lambda(e_3) = \lambda(e_4) = \lambda(e_5) = 1$ and $\lambda(e_2) = -1$. The drawing of G in Figure 1(a) shows that $\text{gcr}(G, \rho, \lambda) \leq 3$. If $\text{gcr}(G, \rho, \lambda) \leq 2$ were true, then e_2 would have to pass through exactly one of the two crosscaps oddly, say $\otimes_1$. Since the ends of e_4 and e_5 alternate with the ends of e_2, both e_4 and e_5 must also pass through $\otimes_1$ oddly. Since e_4 and e_5 are two-sided, they must then also pass through $\otimes_2$ oddly. But then e_4 and e_5 would be parallel (in the sense that their ends do not alternate), contradicting the fact that their ends alternate in the rotation. Hence, $\text{gcr}(G, \rho, \lambda) = 3$. The embedding in Figure 1(b) shows that $\text{dcr}(G, \rho, \lambda) \leq 4$, so we are left with the proof that $\text{dcr}(G, \rho, \lambda) \geq 4$. Suppose, for a contradiction, that G can be realized on a surface with three crosscaps so that every edge passes through each crosscap at most once, and the embedding scheme is (ρ, λ), as specified in Figure 1(a). Then each edge in $\{e_1, e_3, e_4, e_5\}$ passes through an even number of crosscaps. Since none of these edges can be separating (since they would all separate ends of other edges in the rotation), they each pass through two crosscaps. Edge e_2 passes through an odd number of crosscaps. It cannot pass through all three crosscaps, since then all other edges would be parallel to it (as each would share two crosscaps with e_2), but the ends of e_2 alternate with the ends of e_4 and e_5. Hence, e_2 passes through exactly one crosscap, say $\otimes_1$. Since e_3 is parallel to e_2, it must then pass through $\otimes_2$ and $\otimes_3$. Now e_4 and e_5 alternate ends with both e_2 and e_3, so one of them, say e_4, by symmetry, passes through $\otimes_1$ and $\otimes_2$ and e_5 passes through $\otimes_1$ and $\otimes_3$.

Edge	$\otimes_1$	$\otimes_2$	$\otimes_3$
e_2	1	0	0
e_3	0	1	1
e_4	1	1	0
e_5	1	0	1
e_1	0	1	1

Now e_1 is parallel to e_2 and e_3 and passes through two crosscaps, which must therefore be $\otimes_2$ and $\otimes_3$. Now suppose there were such a drawing. Since edges pass through crosscaps at most once, we can think of crosscaps as vertices. But then, there is a path from an end of e_1 to an end of e_3 which passes through $\otimes_2$ and $\otimes_3$ but not through $\otimes_1$. That path now separates the two ends of e_2, since e_2 may only pass through $\otimes_1$. ■

Question 1. Can the construction in Theorem 4 be used to construct for every n a single-vertex graph G with embedding scheme (ρ, λ) so that $n \leq \text{dcr}(G, \rho, \lambda) \leq 3/4\, \text{gcr}(G, \rho, \lambda)$?

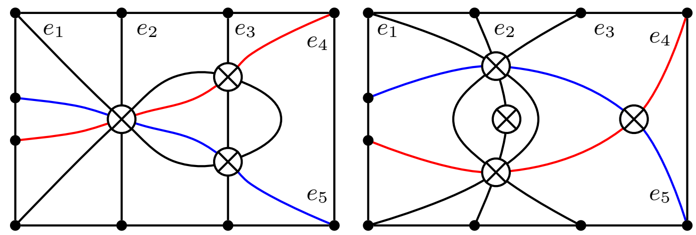

Fig. 1. Graph G with rotation displayed as outer cycle. (a) G embedded in a surface with three crosscaps, requiring e_1 to pass through one crossscap twice. (b) G embedded in a surface with four crosscaps, each edge passing through each crosscap at most once.

5 Nice Embeddings of Higher Genus Graphs

In this section we consider relaxing the restriction on how often each edge may pass through each crosscap. It turns out that increasing the limit to two is sufficient.

Theorem 5. *If a graph is embeddable in a non-orientable surface $\mathcal{S}$, then it can be embedded in $\mathcal{S}$ so that every edge passes through each crosscap at most twice.*

This means, G always has a nearly-degenerate drawing in the plane with at most $\mathrm{gcr}(G)$ crossings, and in which each edge has at most $\mathrm{gcr}(G)$ self-crossings.

By Theorem 4 the theorem is tight if the graph is given with an embedding scheme (which may not be changed), even if the graph consists of a single vertex.

We will concentrate the proof in a more technical lemma, which may be of interest in its own right. For the proof, we need the *Euler genus*, $\mathrm{eg}(G, \rho, \lambda)$ of an embedded graph, which is defined as $1 + |E| - |F|$, where $|E|$ is the number of edges of G and $|F|$ the number of faces in the embedding scheme (ρ, λ) of G (note that this is a purely combinatorial notion). It's tempting to assume that $\mathrm{gcr}(G, \rho, \lambda) = \mathrm{eg}(G, \rho, \lambda)$, but that is not actually true; take, for example, a single vertex with two two-sided edges alternating at the vertex. The Euler genus of this graph is 2, while it requires 3 crosscaps to realize. The following lemma clarifies the relationship.

Recall that an embedded single-vertex graph (G, ρ, λ) is orientable, if $\lambda(e) = 1$ for all $e \in E(G)$.

Lemma 5. *If (G, ρ, λ) is a single-vertex graph with embedding scheme, then it has an embedding in a surface with $\mathrm{eg}(G, \rho, \lambda)$ crosscaps in which every edge uses every crosscap at most twice, unless (G, ρ, λ) is orientable, in which case such an embedding exists in a surface with $\mathrm{eg}(G, \rho, \lambda) + 1$ crosscaps.*

We leave the proof of Lemma 5 to the journal version of the paper. The proof can be viewed as a (more sophisticated) extension of the proof of Theorem 3. Since we allow edges to cross through a crosscap twice, the construction becomes simpler, in that we can process one-sided loops, even if they are not closest.

The new ingredient needed is a technique for dealing with separating loops. For example, consider the embedding scheme described by $\rho(v) = (abbacdcd)$, and $\lambda(b) = -1$, and $\lambda(a) = \lambda(c) = \lambda(d) = 1$, as illustrated in Figure 2(a). The Euler genus of this graph is 3, and a is a separating loop, splitting the graph into two pieces, one of Euler genus 1, and the other of Euler genus 2. The problem now is that the piece of Euler genus 2 is orientable, and hence needs 3 crosscaps to realize by itself. Hence, some care is needed when merging drawings in this case; the solution in this case is shown in Figure 2(b). Details will be found in the journal version.

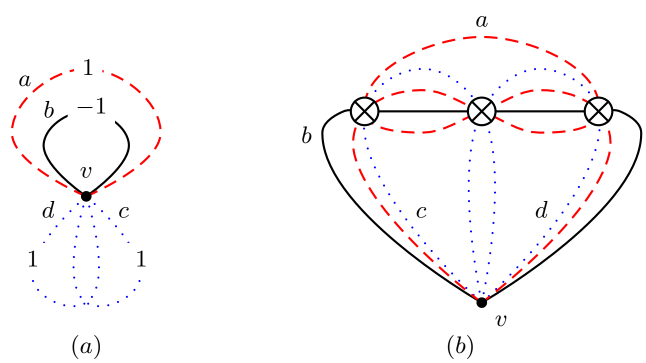

Fig. 2. (a) Embedding scheme with Euler genus 3; edges are a (red/dashed), b (black), and c, d (blue/dotted). (b) Actual embedding of same scheme on surface with three crosscaps, in which every edge passes through every crosscap at most twice.

Proof (of Theorem 5). Fix an embedding of a graph G on a surface with $k = \widetilde{\gamma}(G)$ crosscaps (without loss of generality, we can assume that it is a minimum genus embedding). By Lemma 2 there is a single-vertex graph G' with embedding scheme (ρ, λ) so that $\mathrm{gcr}(G) = \mathrm{gcr}(G', \rho, \lambda)$. It is sufficient to prove the result for G', since an embedding of G' with embedding scheme (ρ, λ) can be turned back into an embedding of G by uncontracting and deleting edges (in case G was not connected). Since these operations can be done close to the single vertex of G', this does not affect how often edges pass through any crosscap. Hence, we can assume that G is given as a graph on a single vertex v with embedding scheme (ρ, λ).

Since (G, ρ, λ) is an embedding on the surface with k crosscaps, $\mathrm{eg}(G, \rho, \lambda) \leq k$. If (G, ρ, λ) is not orientable, then the result follows immediately from Lemma 5. If (G, ρ, λ) is orientable, we apply Corollary 1 to extend (G, ρ, λ) to an embedding scheme (G', ρ', λ') which still embeds in the same surface, and is no longer orientable. Since $\mathrm{eg}(G, \rho, \lambda) \leq \mathrm{eg}(G', \rho', \lambda') \leq k$, and (G', ρ', λ') is not orientable, Lemma 5 gives us an embedding of (G', ρ', λ'), and thereby (G, ρ, λ) in a surface with k crosscaps, in which every edge passes through each crosscap at most twice, completing the proof. ■

The proof of Theorem 5 is entirely combinatorial, so it can be made algorithmic.

Corollary 2. *Determining the degenerate crossing number is* **NP***-complete, even if the graph is cubic.*

Proof. The problem lies in **NP** (since every edge passes through each crosscap at most once we can guess the embedding). On the other hand, Thomassen [13, 9] showed that the non-orientable genus problem is **NP**-complete, even for cubic graphs. For a given cubic graph G, let G' be the result of replacing each edge of G with a path of length $2|E(G)|$, and attaching a (local, planar) gadget to each vertex of degree 2, to ensure that G' is cubic. If G has orientable genus at most k, then, by Theorem 5, there is an embedding in which every edge passes through each of the crosscaps at most twice. Since we can assume that $k \leq |E(G)|$ (e.g. [10]), this implies that G' can be embedded so that every edge passes through each crosscap at most once. In other words, the degenerate crossing number of G' equals the non-orientable genus of G, showing that dcr is **NP**-complete. ∎

6 Open Questions

The main open question which remains is whether $\mathrm{dcr}(G) = \mathrm{gcr}(G)$; one could weaken this question in various ways, and, for example ask whether $\mathrm{dcr}(G) \leq \mathrm{gcr}(G) + c$ for some constant c? Another approach would be to ask whether $\mathrm{dcr}(G) = \mathrm{gcr}(G)$ if we allow a limited number of self-crossings along each edge. Theorem 5 implies that $\mathrm{gcr}(G)$ self-crossings along each edge are sufficient, but can a constant bound be achieved?

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