HANANI-TUTTE AND HIERARCHICAL PARTIAL PLANARITY

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Abstract. We establish a Hanani-Tutte style characterization for hierarchical partial planarity and initiate the study of partitioned partial planarity.

Key words. Hanani-Tutte theorem, hierarchical partial planarity, partitioned partial planarity, graph drawing

AMS subject classifications. 68R10, 05C62

1. Introduction.

Given a graph G whose edge-set has been partitioned into three sets E_1, E_2, E_3, we say that a drawing D of G, is *hierarchically partial planar (with respect to E_1, E_2, E_3)* if no edges of E_1 are involved in any crossings, and edges of E_2 do not cross each other. In other words, if two edges cross in D, one of them must both belong to E_3 and the other to E_2 or E_3. We write $G(E_1, E_2, E_3)$ for G if we want to emphasize the edge-partition. Figure 1 shows a sample hierarchical partial planar drawing.

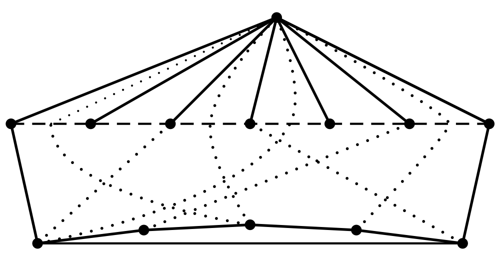

FIG. 1. *A hierchical partial planar drawing with E_1-, E_2- and E_3-edges shown as solid, dashed and dotted, respectively.*

Angelini and Bekos [1] introduced the notion of hierarchical partial planarity as a model in which edges are ordered by "importance" and more important edges are involved in fewer crossings: E_1-edges are crossing-free, and E_2-edges may only cross E_3-edges. They showed that it can be solved in time $O(|V(G)|^3)$ using SPQR-trees (and an intermediate problem which they call facial-constrained core planarity).

We show that hierarchical partial planarity has a Hanani-Tutte style characterization. A Hanani-Tutte style characterization weakens any requirement that a pair of independent edges may not cross to requiring them to cross an even number of times (including not at all). So Hanani-Tutte characterizations work with the *crossing parity* of two edges in a drawing, namely the parity of how often the two edges cross. An edge is called *(independently) even* if it crosses every other (independent) edge in the graph an even number of times. We refer to two (independent) edges crossing oddly as an *(independent) odd pair*.

With this terminology, we can define the Hanani-Tutte version of hierarchical partial planarity: We say a drawing D of $G(E_1, E_2, E_3)$ is $\mathbb{Z}_2$*-hpp* if (i) all edges in E_1 are independently even, and every two independent edges in E_2 cross each other an even number of times. A drawing is *(intersection)-simple* if every two edges intersect

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at most once, counting a shared endpoint.[1]

THEOREM 1.1. *A graph $G(E_1, E_2, E_3)$ has a simple, hierarchically partial planar drawing if and only if it has a $\mathbb{Z}_2$-hpp drawing.*

An immediate consequence—as with (nearly) all Hanani-Tutte style characterizations—is a very simple polynomial-time algorithm for testing hierarchical partial planarity. The reason is that these characterizations can be expressed as the solvability of a linear system of equations over GF(2). The running time is not competitive with the SPQR-algorithm by Angelini and Bekos.

COROLLARY 1.2. *Hierarchical partial planarity can be tested in polynomial time.*

We give some background on Hanani-Tutte in Section 1.1, including a proof of the corollary. The proof of Theorem 1.1 can be found in Section 2.

Angelini and Bekos view hierarchical partial planarity as only one special case of a more general planarity notion. We try to formalize their point of view as *partitioned partial planarity* in Section 3. Section 1.2 establishes some graph drawing context.

1.1. Hanani-Tutte Characterizations. Corollary 1.2 is a special case of a generic form of the Hanani-Tutte theorem suggested in [19]. Given a planarity notion, call it X-planarity, the corresponding Hanani-Tutte variant $\mathbb{Z}_2$-X-planarity is obtained by requiring that any pair of independent edges that are not allowed to cross in an X-planar drawing, cross each other evenly. By definition, X-planarity implies $\mathbb{Z}_2$-X-planarity. The program suggested in [19] is to study for which notions of planarity, X-planarity and $\mathbb{Z}_2$-X-planarity are equivalent.

Examples for which equivalence can be shown include partially embedded planarity [19], level-planarity [10], radial planarity [9, 8], partial planarity [20], some forms of c-planarity [7] and several special cases of simultaneous planarity of two graphs [19].[2] On the other hand, it is known that this generic Hanani-Tutte characterization fails for c-planarity [7] and, thereby, simultaneous planarity of two graphs in general [13].

Given an edge e and a vertex v in a drawing D of a graph G, an (e, v)-*move* is performed as follows: we choose a curve γ connecting a point p on e to v so that γ has only finitely many intersections with the edges in the drawing, and does not pass through a vertex. We then erase e in a small neighborhood of p and reroute it along γ, around v, and back along γ, see Figure 2. An (e, v)-move changes the crossing parity between e and any edge incident to v and affects no other crossing parity; in particular, the effect of an (e, v)-move is independent of the choice of γ.

An (e, v)-move may result in self-intersections of edges; self-intersections can always be resolved locally, as shown in Figure 3, without changing the crossing parity of any pair of edges. From this point on, we will always assume that self-intersections are removed in this way, without mentioning this explicitly.

At the root of the effectiveness of Hanani-Tutte characterizations is the following fact which was rediscovered many times: If we have two drawings D and D' of the same graph, then we can apply a set of (e, v)-moves to D so that the resulting drawing has the same vector of crossing parities between pairs of independent edges as has D' (for a proof, see [18, Theorem 1.18] or [19, Lemma 3.3]; we assume that each drawing has only finitely many intersections).

[1] The drawing in Figure 1 is not simple, two of the edges attached to the top node cross each other. This crossing can be removed by rerouting the two edges involved.

[2] Table 1 in [19] summarizes the results known at the time. We should also mention [6] which does not fit the X-planarity pattern, since it is about approximating embeddings.

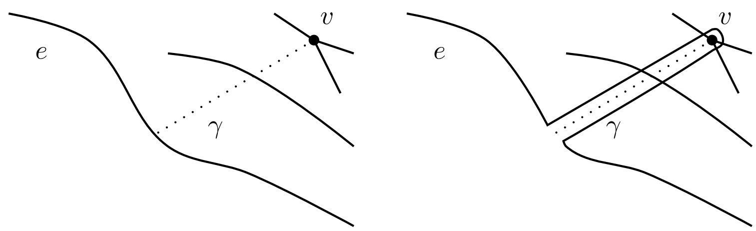

FIG. 2. *Performing an (e, v)-move along γ.*

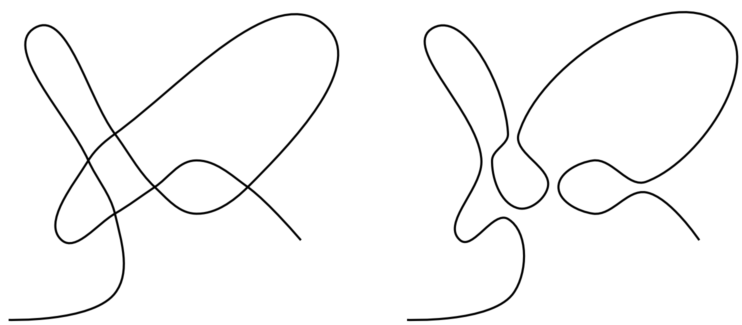

FIG. 3. *Removing self-intersections of a curve by rerouting the curve close to crossings.*

This observation allows us to set up a system of linear equations over GF(2) for testing whether a graph $G(E_1, E_2, E_3)$ has a $\mathbb{Z}_2$-hpp drawing with respect to E_1, E_2, E_3. Suppose D is a drawing of $G(E_1, E_2, E_3)$ (e.g. placing the vertices in convex position). Let $i_D(e, f)$ denote the crossing parity of (e, f) in D. Create 0-1 variable $x_{e,v}$ for every $e \in E(G)$ and $v \in V(G)$, and let HPP(D) be the following system of equations:

$$i_D(st, uv) + x_{st,u} + x_{st,v} + x_{uv,s} + x_{uv,t} = 0 \bmod 2$$

for every $(st, uv) \in (E_1 \times E(G)) \cup (E_2 \times E_2)$. Then, by Theorem 1.1, hierarchical partial planarity is equivalent to the solvability of HPP(D): the set of (e, v)-moves required to turn D into a $\mathbb{Z}_2$-hpp drawing are those for which $x_{e,v}$ is one.

It follows that hierarchical partial planarity can be tested in polynomial time by solving a system of linear equations over GF(2) with $|V||E|$ variables, and $|E|^2$ equations, proving Corollary 1.2. This is not competitive with the SPQR-approach, though the implementation will be much simpler (and much more generic).

1.2. Related Graph Drawing Problems. Given a graph $G = (V, E)$ and a symmetric relation $R \subseteq E^2$ on the edges of G we say that a drawing D of G is a *weak realization* of (G, R) if only pairs of edges in R cross in D (in a *strong realization* exactly the pairs of edges in R cross in D). Weak realizability is powerful enough to capture nearly every other (topological, i.e. non-geometric) graph drawing notion [22, 19]. Unsurprisingly then, it is **NP**-hard [14]. Somewhat surprisingly, it belongs to **NP** [21] in spite of a classical result by Kratochvíl and Matoušek [15] which shows that a weak realization may require an exponential number of crossings. It is easy to construct a (G, R) for which no weak realization is simple.

Since weak realizability is **NP**-hard, we want to identify relations R which lead to useful and tractable variants of the problem. For example, if R is a complete k-partite graph on E (considered as a vertex-set), we obtain a simultaneous planarity problem called SEFE$_k$ (simultaneous embeddability of k graphs with fixed edges). Given a family of k (not necessarily connected) graphs $G_i = (V, E_i)$ on the same vertex set V, we say that the graphs have a *simultaneous embedding (with fixed edges)* if there is a drawing of $G = (V, \bigcup_i E_i)$ in which no two edges belonging to the same graph cross each other. In other words, the induced drawing of G_i is plane for each of the graphs. "Fixed edges" refers to the fact that *public* edges, that is, edges belonging to more than one graph, are drawn the same for all graphs they belong to; *private* edges belong to only one graph.

The SEFE$_3$ problem, simultaneous planarity of three graphs, is known to be **NP**-complete [12], but the computational complexity of SEFE$_2$ is open. The simultaneous planarity of two graphs is a particularly attractive problem to study. While not as universal as weak realizability, it does capture a fair number of other graph drawing problems [19, Figure 2]. A polynomial-time algorithm for SEFE$_2$ would unify a large number of graph drawing algorithms; finding such an algorithm will be hard, however, since SEFE$_2$ generalizes c-planarity, whose computational complexity had been open for twenty-five years before recently being shown polynomial-time solvable by Fulek and Tóth [11].[3]

The SEFE$_k$ problem (for unbounded k) is equivalent to weak realizability [12], so it is not surprising that one can build families of graphs so that any simultaneous planar drawing of these graphs requires an exponential number of crossings (in the number of the graphs).

For the SEFE$_2$ problem it is known that if two graphs G_1, G_2 over the same vertex set have a simultaneous embedding, then any two edges cross at most a constant number of times [3, 5]. It is tempting to conjecture that a positive instance of SEFE$_k$ can always be realized with at most $O(2^k|G|)$ crossings between every pair of edges, but this question seems to be open. We prove a slightly weaker bound for the special case $k = 3$, since we need it for a later application.

THEOREM 1.3. *If three graphs G_1, G_2, G_3 over the same n-vertex set have a simultaneous embedding with fixed edges, then they have such an embedding in which any two edges cross at most $O(n^2)$ times.*

Proof. Let H be the subgraph of $G_1 \cup G_2 \cup G_3$ consisting of all public edges, that is, edges that belong to at least two of the graphs G_1, G_2, and G_3. Fix a simultaneous embedding D of G_1, G_2, G_3, and let $D[H]$ be the drawing of H in D. Any two edges of H belong to at least one common graph, so $D[H]$ is plane (crossing-free). Using a homeomorphism of the plane, we can assume $D[H]$ is a straight-line drawing. By Theorem 1 from [3], we can extend the straight-line drawing of $D[H]$ to a plane drawing of G_i, for each i, so that each edge in $E(G_i) - E(H)$ has at most $72|V(H)| \leq 72n$ bends. Combining the three drawings (possibly perturbing some of the private vertices to avoid overlap), we obtain a simultaneous embedding of G_1, G_2 and G_3. Since every edge consists of at most $72n$ line segments, any two edges cross at most $(72n)^2$ times. ∎

It would be interesting to know whether Theorem 1.3 can be extended to $O(n)$

[3]Bläsius, Fink, and Rutter [2] shortly afterwards improved the running time to $O(n^2)$.

for SEFE_k, or, even better, $O(2^k n)$ as suggested earlier.

Remark 1.4 (Linear Bound—SEFE_2). In the case of two graphs G_1, G_2, we can start with a straight-line embedding of G so that the drawing of H is isomorphic to $D[H]$. Then extending $D[H]$ to a plane embedding of G_2 leads to at most $72|H|$ bends per edge for edges in $E(G_2) - E(H)$, so any two edges of G_1 and G_2 cross at most $O(n)$ times. This same trick does not work for $k = 3$, since we have to add two graphs.

Remark 1.5 (Quadratic Bound—Sunflower SEFE_k). In the *sunflower* variant of SEFE_k every public edge must belong to all k graphs. The construction described in the proof of Theorem 1.3 also works for the sunflower case of SEFE_k, where $k > 3$. So there is an upper bound of $O(n^2)$ crossings for each pair of edges in that case. Can that be improved to $O(n)$?

2. Removing Even More Independently Even Crossings. To prove Theorem 1.1 we would like to follow the usual strategy for Hanani-Tutte theorems: incrementally clean edges of unwanted crossings. Since hierarchical partial planarity generalizes partial planarity we know that this direct approach will not work. The Hanani-Tutte theorem for partial planarity [20] is based on removing independently even crossings [16], which requires modification of the underlying graph. Not surprisingly, we also need to modify the underlying graph for hierarchical partial planarity, as described in Lemma 2.2. As a tool for this step, we need Lemma 2.1 which shows that it is possible to change the crossing parity between two edges (in certain circumstances). Finally, Lemma 2.3 ensures that the final hierarchical partial planar drawing is simple (which is not typically a concern for other drawing notions).

Call an edge $e \in E_1$ *clean* in a drawing of $G(E_1, E_2, E_3)$ if e is free of crossings; an edge $e \in E_2$ is *clean* if it only crosses edges not in $E_1 \cup E_2$ and crosses each such edge at most once.

LEMMA 2.1. *Let D be a drawing of $G(E_1, E_2, E_3)$ in which $F' \subseteq E_1 \cup E_2$ is a set of clean edges that contains all cycle-edges of $G[E_1]$ and so that every edge of F' belongs to a cycle in $G[F']$. Suppose $g \in E_1 - F'$ does not belong to a cycle in $G[E_1]$ and $f \notin E_1 \cup E_2$. Then we can find a drawing D' of G in which the edges of F' are still clean, and the crossing parity between f and g has changed. The only other crossing parity changes may occur between f and edges of $E_2 \cup E_3$.*

Proof. Let $g = uv$, and let U consist of all vertices in $G[E_1 - g]$ that belong to the connected component containing u. Since g does not belong to a cycle in $G[E_1]$, we know that $v \notin U$. We now perform (f, u')-moves for every $u' \in U$. This only affects the crossing parity of f with other edges. Specifically, the crossing parity of f and g changes, since $v \notin U$, and the crossing parity of f with all E_1-edges other than g remains the same; the reason is that every E_1-edge other than g has either both or neither of its endpoints in U. The crossing parity of f with E_2- and E_3-edges may change, but only for those edges with an (exactly one) endpoint in U.

After these moves, F' need no longer be clean, since we may have added crossings between f and edges in F', but we can fix this. Let e be an arbitrary edge in F'. If f crosses e an even number of times (and at least once), we sever all crossings of f with e on both sides of e. If f crosses e an odd number of times (so $e \in E_2$), we sever all but one crossing of f with e. We now have an even number of ends of f on each side of e, so we can pair them up (on each side) and reconnect them, see Figure 4.

This makes e clean, but may result in f consisting of multiple components. One of the components, the *arc*, connects the endpoints of f, and there may be additional

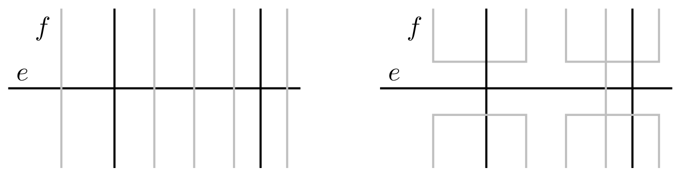

FIG. 4. *Severing crossings of f (gray) with e and reconnecting severed ends (introducing a self-intersection of f which can be removed as in Figure 3).*

components of f which are closed curves. We perform the cleaning process for all e in F'. At the end, all edges in F' are clean, but f may consist of multiple components. If it is possible to reconnect any components without crossing any edges in F', we do so. At this point we want to drop all remaining closed-curve components of f. The only way this could lead to a problem is if after dropping the closed-curve components, the crossing parity of f and some edge $h \in E_1 \cup E_2$ becomes odd. For this to happen, the arc-component of f must cross h, as must (at least) one of the closed-curve components of f. If we cannot reconnect the closed-curve component to the arc-component, then they must be separated by a cycle in $G[F']$. Since h crosses both, it must cross some edge of the cycle oddly. But this can only happen if $h \in E_3$, which is a contradiction. Hence, we can drop all remaining closed-curve components of f. ∎

We say that G' results from *splitting* a vertex v in G if G' contains an edge v_1v_2 so that contracting that edge yields G with $v = v_1 = v_2$.[4] With this definition of vertex split, we can naturally write $E(G) \subseteq E(G')$. Figure 8 shows two examples vertex splits.

The following lemma is a refined version of Lemma 2.3 in [16]. The proof uses similar ideas.

LEMMA 2.2. *Suppose that $G(E_1, E_2, E_3)$ has a drawing D in which all edges of $E_1 \subseteq E(G)$ are independently even, and every two independent edges in E_2 cross each other an even number of times. Then there is a graph $G'(E_1', E_2, E_3)$, which results from G by a sequence of vertex splits, and a drawing D' of G' so that*
(*i*) *edges in E_1' are independently even, and every two independent edges in E_2 cross each other evenly,*
(*iia*) *edges in E_1' that are part of a cycle in $G'[E_1' \cup E_2]$ are clean,*
(*iib*) *edges in E_2 that are part of a cycle in $G'[E_1' \cup E_2]$ are clean,*
(*iii*) *every vertex v that lies on a cycle C in $G'[E_1 \cup E_2]$ has degree at most three.*

In plain English: at the cost of splitting some vertices of G, we can clean those edges in E_1' and E_2 which are cycle edges in $G'[E_1' \cup E_2]$, by (*iia*) and (*iib*). We can ensure that vertices on cycles in $G'[E_1' \cup E_2]$ are incident to at most one non-cycle edge, by (*iii*), and edges in E_1' are still independently even, and edges in E_2 are not involved in independent odd crossings with each other, by (*i*). Any new edges resulting from vertex splits must belong to E_1', because E_1', E_2 and E_3 partition G'. The left illustration in Figure 5 shows the starting situation described by the lemma,

[4] Vertex splits are also often defined as the opposite of merging two vertices which do not have an edge between them.

and the right illustration the modified G' after cleaning the drawing.

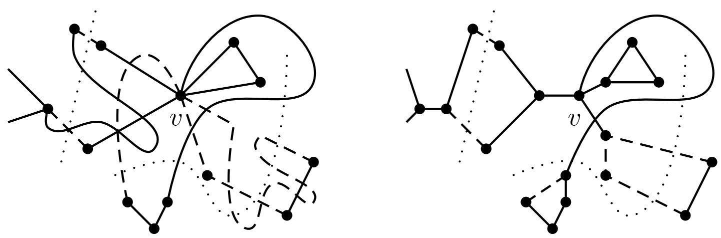

FIG. 5. *A drawing satisfying the assumptions of Lemma 2.2 on the left, with four, uncleaned, cycles in $G[E_1 \cup E_2]$ meeting in a vertex v. The cleaned version of the same drawing is shown on the right. As before, E_1-, E_2-, and E_3-edges are solid, dashed, and dotted (respectively).*

Proof. Letting $E_2 = E_2 - E_1$ if necessary, we can assume that E_1 and E_2 are disjoint. Our first goal is to clean all cycle-edges in $G[E_1 \cup E_2]$. We let F be the set of cycle-edges in $G[E_1 \cup E_2]$ which we have cleaned already. Initially, $F = \emptyset$, and F is trivially clean.

We prove the result by induction on the sum of the cubes of the vertex degrees in G, and, that sum being the same, the number of edges not in F. This induction order allows us to split a vertex of degree $d \geq 4$ into two vertices of degree $d_1 \geq 3$ and $d_2 \geq 3$, since $d_1^3 + d_2^3 < d^3$ for $d_1 + d_2 = d + 2$.

Suppose that *there is a cycle-edge in $G[E_1 \cup E_2]$ which does not belong to F*. Pick a cycle C in $G[E_1 \cup E_2]$ containing such an edge for which $|C \cap E_2|$ is minimal.

If two consecutive edges uv,vw of C cross oddly, we perform a (uv, v)-move, so the two edges cross evenly (and the crossing parity of no pair of independent edges is affected), see the left two illustrations in Figure 6. In this fashion, we can ensure that every two edges of C cross each other evenly (for pairs of independent edges this is part of the assumption). If there is an odd pair vw, vx with $vw \in C$ and $vx \notin C$, we can move vx in the rotation at v so that vw and vx cross evenly (without affecting the crossing parity between vx and the other edge uv in C incident to v). The edges in $C \cap E_1$ are now even, and the edges in $C \cap E_2$ can only cross edges in E_3 oddly.

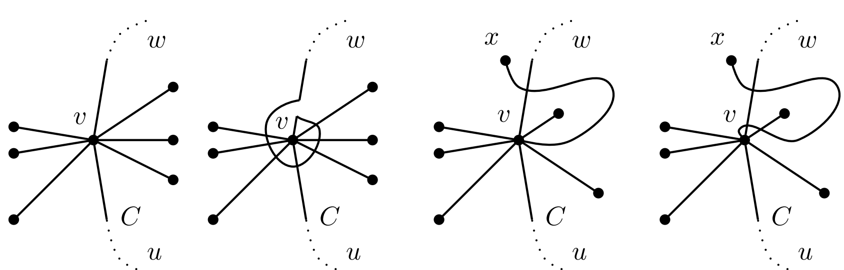

FIG. 6. *An edge uv on the cycle C, and how to make them even with respect to the next edge vw on C (left two illustrations), or with respect to another edge vx not on C (right two illustrations).*

Let e be an edge of C. For every edge f crossing e evenly, we sever all crossings of e with f. For every edge f crossing e oddly, we sever all but one crossing of e with f. Each edge that used to cross e now has an even number of ends on both sides of e. We reconnect these ends pairwise (as we did in Figure 4). This does not change the crossing parity of any pair of edges, but some edges will now consist of multiple curves, one of them, the arc, connecting the endpoints of the edge. We perform this operation for all edges $e \in C$.

At this point we drop all closed-curve components belonging to edges in C. Then all edges of C are clean (by construction), but other edges may still consist of multiple components. We process any remaining closed curves as follows: If we can reconnect a closed-curve component of an edge to the edge's arc-component without crossing F or C, we do so (this may still leave some closed curves), and we do this for all closed-curve components for which it is possible.

Let D^+ be the resulting drawing, and let D^- be D^+ after erasing all remaining closed-curve components. In D^- all edges in $F \cup C$ are clean, but dropping closed components may have created new independent odd pairs. Let f and g be such a pair, that is, f and g are independent edges which cross evenly in D^+ but oddly in D^-. See Figure 7.

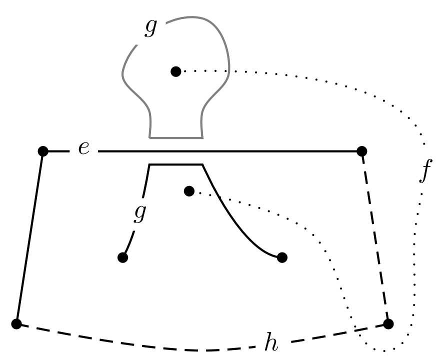

FIG. 7. *After dropping the closed-curve component of g (gray), the arc-components of f and g cross oddly; g was severed when processing edge e on C. Initially, we do not know the types of f, g, and h, but the proof will determine them to be as shown.*

Since any two closed curves cross evenly, at least one of the closed-curve components, say one belonging to g must cross the arc belonging to f in D^+. So the arc-component of g was severed from a closed component (to which it could not be reconnected) when processing some edge $e \in C$. Since both the arc of g and its closed-curve component cross f, we could have tried to reconnect the closed-curve component by following f closely. Since we did not, the crossings of g with f must be separated by a crossing with an edge $h \in F \cup C$. The only way that is possible, is if $h \in E_2$ and $f \in E_3$. Moreover, the independent odd pair only matters (that is, potentially violates the $\mathbb{Z}_2$-hpp condition) if $g \in E_1$. The types are as shown in Figure 7.

We claim that g cannot belong to a cycle C' of E_1-edges. If it did, then this cycle would have fewer E_2-edges (namely none) than the cycle in $F \cup C$ that contains h, so C' would have been picked for processing before that cycle. So C' would already be free of crossings, but we know that g crossed e, which is a contradiction. So g does not belong to a cycle of E_1-edges.

Apply Lemma 2.1 to $g \in E_1 - F$ and f with $F' = F \cup C$. This keeps the edges in $F \cup C$ clean, and the parity of g and f changes, so they cross evenly, as they did in D^-. We do this for all such pairs (g, h), resulting in a drawing in which $F \cup C$ is clean, and all edges in E_1 are independently even, and pairs of independent edges in E_2 cross each other evenly. We can now update F to be $F \cup C$, and we have made progress.

We are therefore in the situation that F *is clean and contains all cycle-edges of* $G[E_1 \cup E_2]$. Suppose there is a cycle C in $G[E_1 \cup E_2]$ and a vertex $v \in V(C)$ so that v has degree larger than 3. If all the edges incident to v lie on the same side of C, we split v into two vertices v_1 and v_2, connected by a crossing-free edge v_1v_2 and with v_2 incident to the edges v was incident on (other than the edges of C). The vertex split decreases the sum of the degrees cubed, so we can apply induction to $G'(E_1', E_2, E_3)$, where $E_1' = E \cup \{v_1v_2\}$, to obtain the result, see the left half of Figure 8.

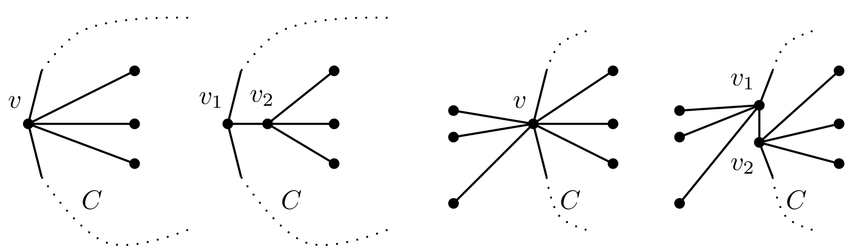

FIG. 8. *Splitting v on C.*

We can therefore assume that v is incident to edges on both sides of C. We want to split v into two vertices v_1, v_2 connected by a new edge v_1v_2, with v_1 taking the edges incident to C from the outside, and v_2 the edges attaching to C from the inside, see the right half of Figure 8.

This move decreases the sum of the degrees cubed, but it may introduce a new independent odd pair. This happens if v is incident, on opposite sides of C, to two edges f and g that cross oddly. At least one of these edges, say f, has to cross C (so f and g can cross). This implies that $f \notin E_1 \cup E_2$. For the crossing parity of f and g to matter then, we must have $g \in E_1$. Now g cannot be a cycle-edge of $G[E_1 \cup E_2]$, otherwise it would be free of crossings. Hence, we can apply Lemma 2.1 with $g \in E_1 - F$, f and $F' = F$ to change the crossing parity of f and g. We do this for all such pairs of edges at v. At the end, we have a drawing in which splitting v as described above does not result in a new independent pair of edges that matters, and we are done by applying induction to $G'(E_1', E_2, E_3)$, where $E_1' = E \cup \{v_1v_2\}$. ∎

One final lemma allows us to make a hierarchical partial planar drawing simple. We will generalize this result in Lemma 3.6.

LEMMA 2.3. *If $G(E_1, E_2, E_3)$ has a hierarchically partial planar drawing, then it has a simple, hierarchically partial planar drawing.*

Proof. Fix a hierarchical partial planar drawing of $G(E_1, E_2, E_3)$. Suppose an edge $f \in E_3$ intersects an edge $e \in E_2$ more than once. We sever all crossings of e with f. If e and f are independent, we remove all pieces of e except the two half-arcs containing its endpoints. We then reconnect the severed ends of the two half-arcs by following f closely, see the left half of Figure 9. If e and f share a common endpoint v, we remove all pieces of e except the half-arc not containing v. We then reconnect the severed end of the other half-arc to v by following f closely, see the right half of Figure 9.

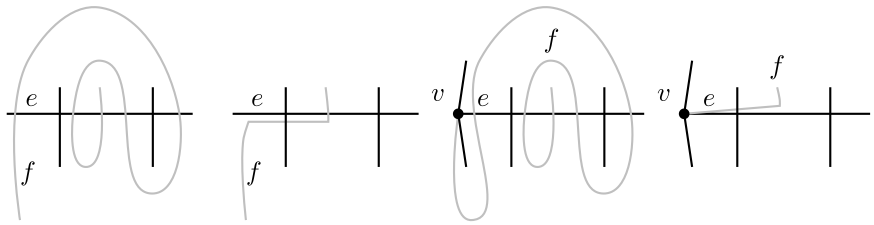

FIG. 9. *Reducing the number of crossings between e and f.*

316 In either case, we reduce the number of intersections between e and f by at least one (since they end up crossing at most once, and this only happens if they crossed more than once before). As a result, the total number of crossings between edges in E_2 and E_3 decreased strictly. Hence, if we repeat this process, we will eventually end up with all edges of E_2 being clean. Two edges $e, f \in E_3$ may still intersect each other more than once. Then there must be subarcs $\gamma_e \subseteq e$ and $\gamma_f \subseteq f$ that have the same endpoints (two crossings, or a crossing and a shared endpoint of e and f); to see this, let γ_e be a shortest subarc of e connecting two intersections of e with f, and let γ_f be the subarc of f connecting the same two intersections. Then γ_e and γ_f do not intersect except for at their shared endpoints. We can now flip γ_e and γ_f, that is, we route e along γ_f and e along γ_e, see Figure 10; the left half illustrates the case of two crossings, the right half the case of a crossing and a shared endpoint.

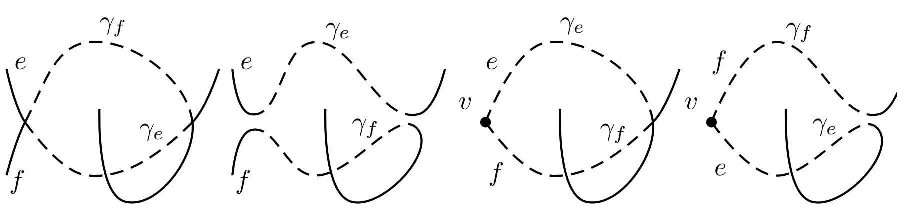

FIG. 10. *Rerouting arcs γ_e and γ_f.*

This rerouting strictly reduces the number of crossings between E_3-edges (and does not increase the number of crossings with E_2-edges). We conclude that after a finite number of steps, any two E_3-edges intersect at most once. ∎

With these three lemmas we can complete the proof of our main result.

Proof of Theorem 1.1. In a hierarchical partial planar drawing, edges in E_1 are even, and edges in E_2 cross each other evenly (namely not at all), so we only have to prove that the Hanani-Tutte condition is sufficient.

Suppose we are given a drawing D of G in which all edges of E_1 are independently even, and independent edges in E_2 cross each other evenly. By Lemma 2.2 we can perform a sequence of vertex splits on G to obtain a graph $G'(E_1', E_2, E_3)$, and a drawing D' of G' satisfying the conditions $(i) - (iii)$ stated in the lemma. Let F be the set of cycle-edges in $G'[E_1' \cup E_2]$. By condition (ii), all edges in F are clean in D'. In particular, edges in $F \cap E_1'$ are free of crossings, and edges in $F \cap E_2$ only cross edges in E_3.

We start with the plane embedding of $G'[F]$. Let e be an edge in $E'_1 \cup E_2 - F$. The endpoints of e belong to the same face boundary of $G'[F]$, since e connects its endpoints in D' without crossing edges in F. We can therefore add e to the embedding without creating any crossings, and without changing which vertices belong to a face boundary (since e does not belong to a cycle in $E'_1 \cup E_2$). Repeating this for all edges in $E'_1 \cup E_2 - F$ gives us a plane embedding of $G'[E'_1 \cup E_2]$ in which any two vertices that belonged to the same face boundary in the plane embedding of $G'[F]$ still do so.

Let $e \in E_3$. We have a drawing of e in D' in which it connects its endpoints without intersecting any edge in $E'_1 \cap F$. Hence, the endpoints of e lie on the same face boundary of the plane embedding of $G'[E'_1 \cap F]$, and, therefore $G'[E'_1]$, since adding edges in $E'_1 - F$ did not change which vertices lie on the boundary of a face. We can therefore add e to the drawing, so that it does not cross any edge in E'_1, though it may cross edges in E_2 (any number of times). We do this for all edges e not in $E'_1 \cap E_2$. In the resulting drawing, there may be multiple crossings among edges not in $E'_1 \cup E_2$ and between edges in $E'_1 \cup E_2$ and E_2. Lemma 2.3 now gives us a simple drawing of $G'(E'_1, E_2, E_3)$ in which edges of E'_1 and E_2 are clean. Specifically, edges in $E'_1 - E_1$ are free of crossings, and we can contract them, to obtain the required simple, hierarchical partial planar drawing of $G(E_1, E_2, E_3)$. ∎

3. Partitioned Partial Planarities.

Angelini and Bekos [1] suggest that hierarchical partial planarity is just one of several planarity variants that can be obtained by partitioning the edge set of a graph into types and specifying which types of edges may intersect. We try to capture their idea a bit more formally by introducing the notion of *partitioned partial planarity.*

For a graph $G(E_1, \ldots, E_k)$ a notion of (k-)partitioned partial planarity is defined by specifying a symmetric relation R over $\{1, \ldots, k\}$, where $R(i, j) = 1$ means that edges in E_i may cross edges in E_j, and 0 that they may not. Partitioned partial planarity refines weak realizability (which is the special case where each edge-set contains a single edge).

Since R is symmetric, we can write R as the upper triangle of the matrix representing R. E.g. the relation R for hierarchical partial planarity is

$$\begin{array}{|ccc} 0 & 0 & 0 \\ \hline & \multicolumn{1}{|c}{0} & 1 \\ \cline{2-3} & & \multicolumn{1}{|c}{1} \\ \cline{3-3} \end{array} .$$

For inline display we abbreviate this to $|000|01|1$. We say two edge types i and j are *equivalent* if $R(i, k) = R(j, k)$ for all k and $R(i, i) = R(j, j)$. If we have two equivalent edge types, we can merge them into a single edge-type without changing the underlying problem. We therefore define two partitioned partial planarity variants as *equivalent* if they are the same up to merging equivalent edge types and relabeling edge types. A variant is *monotone* if it is equivalent to a monotone matrix, that is, a matrix in which the entries in each row and column are non-decreasing. An edge type i is *trivial* if $R(i, j) = 1$ for all j, and a hierarchical planarity variant is *trivial* if it contains a trivial edge type. We can always eliminate a trivial edge type without affecting the complexity of the planarity problem.

3.1. The Case of Small k.

To get a sense of the descriptive power of partitioned partial planarity we have a closer look at the variants we obtain for k up to 3. Our list eliminates equivalent variants, so, for example, we will not see $|00|0$ because it is equivalent to $|0$. Also, we do not include any variants which contain trivial edge-types, so we will not include $|1$, which is trivial. In some of these cases, the crossing minimization problem may be of independent interest. For example, the crossing minimization problem for $|1$ amounts to the standard crossing number, and while $|01|1$ just expresses the planarity of $G[E_1]$, the corresponding crossing minimization problem has not been studied as far as I know; it asks for a drawing of G with the smallest number of crossings for which $G[E_1]$, by itself, is planar. (In comparison, the variant in which the plane embedding of $G[E_1]$ is given and fixed, is widely investigated in the crossing minimization literature.)

For $k = 1$ there is only one variant, $|0$, which is standard planarity.

For $k = 2$, we have $|00|1$, which is partial planarity, the special case of hierarchical partial planarity in which $E_2 = \emptyset$. There is a Hanani-Tutte characterization [20] and a linear-time algorithm [4].

There are two non-monotone variants for $k = 2$. The first is $|01|0$; this captures the SEFE$_2$ problem for two edge-disjoint graphs, which is equivalent to both $G[E_1]$ and $G[E_2]$ being planar, so linear-time testable. The Hanani-Tutte characterization consists of two separate planarity problems.

The second non-monotone variant is $|10|1$, which is trivial (embed all vertices along a line, and draw edges of E_1 above, and edges of E_2 below the line).

For $k = 3$, there is one monotone variant, $|000|01|1$, which we already identified as hierarchical partial planarity. We established a Hanani-Tutte characterization in this case, and there is a cubic-time algorithm by Angelini and Bekos [1].

Our first non-monotone variant is $|011|01|0$, which is the simultaneous planarity of three disjoint graphs $G[E_1]$, $G[E_2]$, $G[E_3]$, and is equivalent to each of these graphs being planar.

Next, $|000|01|0$ is the variant that asks whether $G[E_1 \cup E_3]$ and $G[E_1 \cup E_2]$ have plane embeddings which are isomorphic on $G[E_1]$. This is just the simultaneous planarity problem for two graphs. We know that there is no Hanani-Tutte characterization in this case [13], and the computational complexity of the problem is famously open.

There are several variants extending the trivial $|10|1$. Variant $|011|10|1$ is still trivial using a similar construction as in the $k = 2$ case: start with a plane embedding of $G[E_1]$ with all vertices of V on a line; edges of $G[E_2]$ go above the line, edges of $G[E_3]$ below.

Variants $|010|10|1$, $|000|10|1$ and $|010|00|01$ also extend $|00|1$, so they are not trivial, but we have not been able to determine what their complexity is, whether they express some natural planarity notion, and whether there is a Hanani-Tutte theorem for these variants.

Similarly, $|100|10|1$ extends $|10|1$ (without extending partial planarity), and is non-trivial; for example $K_{3,3}$ with each of the E_i being a $K_{1,3}$, is not realizable in this variant, since every drawing of a $K_{3,3}$ must contain a crossing between two independent edges, by the strong Hanani-Tutte theorem for planarity, see, for example, [18, Theorem 1.1]. This variant is a very natural (anti)-planarity notion, but it appears to be unstudied, and its complexity is open.

Finally, $|011|00|1$ is equivalent to $G[E_1]$ being planar, and $G[E_2 \cup E_3]$ being partial planar, with $G[E_2]$ being crossing-free. We simply fix a partial planar drawing of $G[E_2 \cup E_3]$ in which $G[E_2]$ is free of crossings, and add a planar drawing of $G[E_1]$ on the same vertex set.

We leave the small cases with an enumerative question.

Question 3.1. How many non-equivalent, non-trivial partitioned partial planarity variants are there for each k? We saw that the first three values in this list are $(1, 3, 9)$, and a computer simulation suggests the next values for $k = 4$ and $k = 5$ are $(43, 285)$. The sequence $(1, 3, 9, 43, 285)$ does not occur in OEIS [23].

3.2. Observations and Questions. As we saw in the previous section, partitioned partial planarity is already hard to handle for $k = 3$. Nevertheless, we wonder whether there is a dichotomy theorem.

Question 3.2. Is it true that partitioned partial planarity is always either polynomial time solvable or **NP**-complete for a fixed R? If so, can we effectively tell which based on R?

We collect some further observations and questions suggested by our short survey in the previous section.

3.2.1. The Monotone Case. The cases up to $k = 3$ suggest that there is only one monotone variant for each k (unless we allow trivial types, in which case there are two), and this is true.

THEOREM 3.3. *There is only one non-trivial, monotone k-partitioned partial planarity variant for each k (up to equivalence).*

Based on this it makes sense to apply the term hierarchical partial planarity introduced by Angelini and Bekos for $k = 3$ for arbitrary $k > 3$. We also write k-hierarchical partial planarity.

Proof. Let R be a monotone partitioned partial planarity variant over k edge-types, so we can assume that R is monotone. The first row cannot contain a 1, since this would lead to a trivial edge-type, hence the first row (and column) consists entirely of zeroes. Every row can contain the pattern 01 at most once, and two rows cannot both contain the pattern 01 in the same consecutive columns (otherwise they'd be equivalent). This implies that each row must contain at least one additional 1 compared to the previous row, and, since the matrix is symmetric, that it contains exactly one additional 1, leading to an R in which all entries on or above the anti-diagonal are 0 and all other entries 1. ◻

The monotone variants were also isolated by Angelini and Bekos as worthy of further study; they suggested that they may form a tractable special case of weak realizability.

Question 3.4. Is k-hierarchical partial planarity polynomial-time recognizable for each fixed k? What about unbounded k?

Question 3.5. Is there a Hanani-Tutte theorem for k-hierarchical partial planarity for $k > 3$?

We showed, in Lemma 2.3, that a k-hierarchically partial planar graph always has a simple realization for $k = 3$. This turns out to be true for arbitrary k.

LEMMA 3.6. *If a graph is k-hierarchically partial planar (for arbitrary k), then it has a simple hierarchically partial planar realization.*

Without monotonicity we cannot guarantee simple realizations, as we will see in the next section.

Proof. Suppose $G(E_1, \dots, E_k)$ has a hierarchically partial planar drawing D; let R be the corresponding relation.

Let $c_D(i,j)$ be the total number of crossings between edges of E_i and E_j. We can choose D such that the sequence $(c_D(i,j))_{1\leq i\leq j\leq k}$ is minimal, where indices (i,j) are arranged in lexicographic order.

Pick a smallest (i,j) in that order so that $R(i,j)$ and there are edges $e \in E_i$ and $f \in E_j$ that intersect more than once in D. As we saw in the proof of Lemma 2.3 there are subarcs $\gamma_e \subseteq e$ and $\gamma_f \subseteq f$ that have the same endpoints (two crossings, or a crossing and a shared endpoint of e and f).

Suppose $i = j$. We can detour e along γ_f and f along γ_e (as in Figure 10). This decreases $c_D(i,i)$ by at least one. Since $i = j$, no other value of the sequence changes, so this contradicts the choice of D. (Note that the detour may introduce self-crossings of arcs, but those can be removed locally as before, see Figure 3.)

We therefore have $i < j$. Let m_e and m_f be the smallest ℓ such that there is an edge $g \in E_\ell$ intersecting γ_e and γ_f, respectively. By the case we are in, we have $m_f \leq i$. If $m_f < m_e$, we detour γ_f along γ_e (without moving γ_e). This decreases $c_D(m_f, j)$, contradicting the choice of D. Hence $m_e \leq m_f \leq i$. Since R is monotone, this means we can detour γ_e along γ_f. We can also detour γ_f along γ_e. Let $c_D(\gamma,\ell)$ denote the number of crossings in D between an arc γ and edges of type ℓ. If $c_D(\gamma_e,\ell)$ and $c_D(\gamma_f,\ell)$ differ for some ℓ with $m_e \leq \ell \leq i$, we pick the smallest ℓ for which they differ, and detour the arc with the larger value along the arc with the smaller value. This strictly decreases $c_D(\ell,j)$ without increasing any values that precede (ℓ,j) lexicographically, contradicting the choice of D. We conclude that $c_D(\gamma_e,\ell) = c_D(\gamma_f,\ell)$ for all ℓ with $m_e \leq \ell \leq i$. We can then detour γ_f along γ_e strictly decreasing $c_D(i,j)$ by at least one, without changing any values that precede (i,j). Again this contradicts the choice of D. ∎

3.2.2. Non-Monotone Variants. We turn to the richer world of non-monotone partitioned partial planarity. The descriptive richness leads to an increased complexity of the resulting problems. It is known that a weak realization of a graph may require an exponential number of crossings [15], which implies that edges may have to cross more than once. And we can force dependent edges to cross, even for |001|00|0, using a standard construction.[5]

While we do not yet know whether k-hierarchical partial planarity is always polynomial-time solvable, we do know that non-monotone variants are not (unless $\mathbf{P} = \mathbf{NP}$).

LEMMA 3.7. *SEFE$_k$ can be expressed as a $(2^k - 1)$-partitioned partial planarity problem.*

Proof. For a SEFE$_k$ problem we are given k graphs $G_1, \ldots, G_k$ over the same vertex set V. With that let $G = (V,E)$, where $E = E(G_1) \cup \cdots \cup E(G_k)$. We partition E into edge-sets $E_I = \bigcap_{i\in I} E(G_i) \cap \bigcap_{i\notin I} \overline{E(G_i)}$, where the index I ranges over all $2^k - 1$ non-empty subsets of $\{1,\ldots,k\}$. Edges in E_I and E_J belong to a common graph if and only if $I \cap J \neq \emptyset$. We can therefore let $R(I,J) = 0$ if $I \cap J \neq \emptyset$ and 1 otherwise. Then $G_1, \ldots, G_k$ have a simultaneous embedding with fixed edges if and only if G can be realized with the given R. ∎

Since SEFE$_3$ is **NP**-complete [12], it follows that k-partitioned partial planarity is **NP**-complete for $k \geq 7$.

[5] An example can be based on the marginal illustration for the entry "local crossing number" in [17].

Question 3.8. What is the smallest k for which k-partitioned partial planarity is **NP**-complete?

This may be a tricky question, since showing that $k > 3$ would require showing that SEFE_2 is polynomial-time solvable.

The variant $|100|10|1$ generalizes naturally by letting R be the identity matrix. This leads to an **NP**-complete problem.

LEMMA 3.9 (The Identity Variant). *Partitioned partial planarity for $R = I$ is* **NP***-complete (for unbounded k).*

The proof translates weak realizability into the $R = I$ variant. For this we need an **NP**-complete special case of weak realizability which can be realized with a polynomial number of crossings. By Theorem 1.3 we can work with SEFE_3.

Proof. We reduce from SEFE_3 which we know to be **NP**-complete [12]. Let G_1, G_2, G_3 be three graphs on the same n-vertex set V; also, let $G = G_1 \cup G_2 \cup G_3$. By Theorem 1.3 if G_1, G_2 and G_3 have a simultaneous embedding with fixed edges, then they have such an embedding with at most cn^2 crossings between any pair of edges, for some integer $c > 0$.

We need to build an edge-partitioned graph H. To simplify the presentation we will describe the partition of the edges of H as a coloring (rather than a numerical labeling). We work with the set of colors $\Sigma = \{\sigma(e, f) : e, f \in E(G)\}$, where $\sigma(e, f) = \sigma(f, e)$ is a unique color assigned to the pair of edges (e, f). Then $|\Sigma| = \binom{m}{2}$ where $m = |E(G)|$.

We start with $V(H) = V$, and no edges. For any edge $e \in G$ let $(f_1, \ldots, f_\ell)$ be the list of all edges that e may cross in a simultaneous embedding of G. We create a path P_e of length $cn^2\ell$ between the endpoints of e and color its edges according to the colors in the list $(\sigma(e, f_1), \ldots, \sigma(e, f_\ell))^{cn^2}$. Two paths P_e and P_f can only cross if they share a color, which must be $\sigma(e, f)$, so this only happens if e and f are allowed to cross in G. Moreover, since we can assume that G has a simultaneous embedding in which every two edges cross at most cn^2 times and there are at most $\ell \leq m$ edges crossing any edge, the path P_e between endpoints e is sufficiently long to accommodate all possible crossings (in any order that they may occur in). ∎

Strictly speaking, Lemma 3.9 is not about a single partitioned partial planarity variant, but about a family of them. We believe that the proof can be adapted to show that the problem remains **NP**-complete for a fixed k. To that end, the paths P_e need to be replaced by (narrow) grids which are colored by a finite set of repeating colors in such a way that only grids that belong to edges that may cross, cross each other, and some care needs to go into attaching the grids to a vertex. We leave it to a more adventurous reader to work out the details. We estimate that the resulting k will be less than a hundred.

Question 3.10. What is the smallest k for which the identity variant is **NP**-complete? What is the computational complexity of $|100|10|1$?

Acknowledgments. I would like to thank the referees for many suggestions that improved the presentation of the paper.

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