On the Induced Matching Problem☆

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Abstract

We study extremal questions on induced matchings in certain natural graph classes. We argue that these questions should be asked for twinless graphs, that is graphs not containing two vertices with the same neighborhood. We show that planar twinless graphs always contain an induced matching of size at least $n/40$ while there are planar twinless graphs that do not contain an induced matching of size $(n + 10)/27$. We derive similar results for outerplanar graphs and graphs of bounded genus. These extremal results can be applied to the area of parameterized computation. For example, we show that the induced matching problem on planar graphs has a kernel of size at most $40k$ that is computable in linear time; this significantly improves the results of Moser and Sikdar (2007). We also show that we can decide in time $O(91^k + n)$ whether a planar graph contains an induced matching of size at least k.

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1. Introduction

A matching in a graph is an *induced matching* if it occurs as an induced subgraph of the graph. Determining whether a graph has an induced match-

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ing of size at least k is NP-complete and remains so even if restricted to bipartite graphs of maximum degree 4, planar bipartite graphs, and 3-regular planar graphs (see [6] for a detailed history). Moreover, approximating a maximum induced matching is difficult: the problem is APX-hard, even for $4r$-regular graphs, for all $r \geq 1$ [6, 17].

There are several classes of graphs for which the problem turns out to be polynomial time solvable, for example chordal graphs and outerplanar graphs (see [6] for a survey and [11] for the result on outerplanar graphs).

In terms of the parameterized complexity of the induced matching problem on general graphs, it is known that the problem is $W[1]$-hard [12]. Hence, according to the parameterized complexity hypothesis, it is unlikely that the problem is *fixed-parameter tractable*, that is, solvable in time $O(f(k)n^c)$ for some constant c independent of k.

Very recently, Moser and Sikdar [11] considered the parameterized complexity of PLANAR-IM: finding an induced matching of size at least k in a planar graph. They showed that PLANAR-IM has a *linear problem kernel*, but left the constant in the kernel size undetermined. Their result automatically implies that the problem is fixed-parameter tractable.

In the current paper we take a combinatorial approach to the problem, establishing lower and upper bounds on the size of induced matchings in certain graph classes. In particular, an application of our results to PLANAR-IM gives a significantly smaller problem kernel than the one given in [11]. We also use our results to give a practical parameterized algorithm for PLANAR-IM that can be extended to graphs of bounded genus and could be used as a heuristic for general graphs.

Let us consider the induced matching problem from the point of view of extremal graph theory: How large can a graph be without containing an induced matching of size at least k? Dense graphs such as K_n and $K_{n,n}$ pose an immediate obstacle to finding a meaningful answer to this question, but they can be eliminated easily by restricting the maximum or the average degree of the graph. Indeed, for strong edge colorings the maximum degree restriction is common in the literature [8, Section 12.21]. A *strong edge coloring* with k colors is a partition of the edge set into at most k induced matchings [15]. A greedy algorithm shows that graphs of maximum degree Δ have a strong edge chromatic number of at most $2\Delta(\Delta - 1) + 1$, and Δ is an immediate lower bound. If we are only interested in a large induced matching however, we might not need to restrict the maximum degree. On the other hand, bounding only the average degree of a graph allows pathological examples such as $K_{1,n}$, which has average degree less than 2 but only a single-edge induced matching. The $K_{1,n}$ example illustrates another obstacle to a large induced matching: *twins*. Two vertices u and v are said to be twins if $N(u) = N(v)$. At most one of u and v can be an endpoint of an edge in an induced matching and if one of them can, either can. Thus, from the extremal point of view (and since twins can be recognized and eliminated efficiently) we should study the induced matching problem on graphs without twins. Twinlessness does not allow us to drop the bounded average degree requirement however, as shown by removing a perfect matching from $K_{n,n}$, which yields a twinless graph with a maximum induced matching of size 2.

We begin by studying twinless graphs of bounded average degree. Those graphs might still not have large induced matchings since they could contain very dense subgraphs (Remark 3.4 elaborates on this point). One way of dealing with this problem is to extend the average degree requirement to all subgraphs. In Section 3 we see that a slightly weaker condition is sufficient, namely a bound on the chromatic number of the graph. We show that a graph of average degree d and bounded chromatic number contains an induced matching of size $\Omega(n^{1/\lceil d \rceil})$.

While we cannot expect to substantially improve the dependency on the average degree of this result in general (Remark 3.3), we do investigate the case of planar graphs and graphs of bounded genus, for which we can show the existence of induced matchings of linear size. Indeed, a planar twinless graph always contains an induced matching of size $n/40$. We also know that this bound cannot be improved beyond $(n + 10)/27$ (Remark 4.12). Planar graphs and graphs of bounded genus are discussed in Section 4.

We next investigate the case of outerplanar graphs: an outerplanar graph of minimum degree 2 always contains an induced matching of size $n/7$ (even without assuming twinlessness), and this result is tight (Section 5). Our bounds fit in with a long series of combinatorial results on finding sharp bounds on the size of induced structures in subclasses of planar graphs (for example, [1, 7, 13, 14]).

We also use our combinatorial results to obtain fixed-parameter algorithms for the induced matching problem. For example, we show that PLANAR-IM can be solved in time $O(91^k + n)$ by a reasonably practical algorithm, while—on the more theoretical side—there is an algorithm deciding it in time $O(2^{159\sqrt{k}} + n)$ using the Lipton-Tarjan [10] separator theorem. Both results easily extend to graphs of bounded genus.

2. Preliminaries

In this paper, graphs are finite and have no loops or multiple edges, unless we specify otherwise. Our terminology and definitions generally agree with West [16].

2.1. Structure of Graphs

For a graph G, $V(G)$ and $E(G)$ are the sets of vertices and edges of $Gn(G) = |V(G)|$ and $e(G) = |E(G)|$ are the number of vertices and edges in G. A graph with one vertex is *trivial*. For a vertex v, we let $N(v)$ be the set of vertices adjacent to v. The *degree* of a vertex v, $\deg(v)$, is the number of edges incident to v in G; $\deg_H(v)$ is the degree of v in a subgraph H of G. $G - v$ is obtained from G by removing $v \in V(G)$ and its incident edges, and $G - e$ (resp. $G + e$) is obtained from G by removing (resp. adding) the edge e.

A *hypergraph* $\mathcal{H} = (V, E)$ consists of a *vertex set* $V = V(\mathcal{H})$ and an *edge set* $E = E(\mathcal{H})$ so that $e \subseteq V$ for every $e \in E$. If E is allowed to be a multiset (elements can repeat) we call $\mathcal{H}$ a *multihypergraph*.

A *matching* in a graph G is a set of edges M such that no two edges in M share the same endpoint. The *size* of a matching is its cardinality. A matching M is said to be an *induced matching* if the subgraph induced by the vertices in M contains only the edges of M. An induced matching M is a *maximum induced matching* if M has the maximum size among all induced matching in the graph. We let $\mathrm{mim}(G)$ be the size of a maximum induced matching in a graph G.

The *blocks* of a graph G are its maximal 2-connected subgraphs, its cut-edges, and its isolated vertices. Two blocks may only intersect at a cut-vertex of G. The *block-cutpoint tree* of a connected graph G is the tree whose vertices are the blocks and cut-vertices of G, with edges from each cut-vertex to the blocks that contain it. A connected graph that is not 2-connected has a nontrivial block-cutpoint tree; its *leaf blocks* are its blocks that are leaves in its block-cutpoint tree. In such a graph it is easy to find a cut-vertex which is in at most one non-leaf block, by deleting all leaves from the block-cutpoint tree and selecting a vertex of degree at most 1 in the remaining graph.

2.2. Graphs in Surfaces

A graph is *planar* if it can be drawn in the plane without edge intersections (except at the endpoints). A *plane graph* has a fixed drawing. Each maximal connected region of the plane minus the drawing is an open set;

these are the *faces*. One is unbounded, called the *outer face*. An *outerplane graph* is a plane graph for which every vertex is incident to the outer face; and *outerplanar graph* is a graph that has such a plane embedding. Outerplanar graphs are precisely the graphs that have no K_4-minor or $K_{2,3}$-minor (analogous to Wagner's characterization of planar graphs). In a 2-connected outerplane graph, the outer face is bounded by a Hamiltonian cycle, and the other edges are *chords* of the cycle.

The *dual graph* G^* of a plane graph G is a plane graph (allowing multiple edges) whose vertices are the faces of G, and for each edge e in G there is an edge e^* in the dual graph between the faces incident to e in G. The boundary of a face in a connected graph is a closed walk; it's an *ℓ-face* if the walk has length ℓ.

If we remove the vertex of G^* that is the outer face of G we obtain the *weak dual* of G. The weak dual of an outerplane graph is a forest, and the weak dual of a 2-connected outerplanar graph is a tree. A leaf in the weak dual of a 2-connected outerplanar graph is a *leaf face*.

A graph has *genus g* if it can be drawn on a surface of genus g (a sphere with g handles) without intersections. We say a (multi)hypergraph $\mathcal{H}$ is *embeddable in a surface* if the bipartite incidence graph obtained from $\mathcal{H}$ by replacing each of its edges by a vertex adjacent to all the vertices in the edge is embeddable in that surface. In particular, this definition allows us to speak of a *planar (multi)hypergraph* or a *(multi)hypergraph of genus g*.

A graph H is a *minor* of G, written $H \preceq G$ if H can be obtained from a subgraph of G by contracting edges. Planar graphs and graphs of genus at most g are easily seen to be downward closed under minors.

2.3. Parameterized Complexity

A *parameterized problem* Q is a set of instances of the form (x, k), where x is the input instance, and k is a positive integer called the *parameter*. A parameterized problem Q is said to be *fixed-parameter tractable* [4] if there is an algorithm that solves Q in time $f(k)|x|^c$, where c is independent of k. If (x, k) is an instance of a parameterized problem Q, then by *kernelizing* the instance (x, k), we mean applying a polynomial time preprocessing algorithm on (x, k) to construct another instance (x', k') of Q, called the *kernel* of (x, k), such that (1) $k' \leq k$; (2) the kernel size $|x'|$ of x' is bounded by a function of k'; and (3) a solution for (x, k) can be constructed in polynomial time from a solution for (x', k'). It has been shown that a parameterized problem is fixed-parameter tractable if and only if the problem is kernelizable [5].

3. Induced Matchings in Graphs of Bounded Average Degree

We will show that twinless graphs of average degree d and bounded chromatic number contain induced matchings of size $\Omega(n^{1/\lceil d\rceil})$. At the core of the proof is a combinatorial result due to Füredi and Tuza [9, Theorem 9.13]. A *system of strong representatives* of a set system $\mathcal{F}$ is a family $(x_F)_{F\in\mathcal{F}}$ such that $x_F \in F - \bigcup_{F'\neq F} F'$ for all $F \in \mathcal{F}$.

Lemma 3.1 (Füredi and Tuza, 1985)**.** *If $\mathcal{F}$ is a collection of at least $\binom{s+\ell}{\ell}$ sets of size at most s, then there is a collection $\mathcal{F}' \subseteq \mathcal{F}$ of size at least $\ell+2$ which has a system of strong representatives.*

Theorem 3.2. *A nontrivial twinless graph G on n vertices with $\chi(G) \leq k$ and average degree d must contain an induced matching of size at least*

$$\lceil d\rceil \left(\frac{1}{e}\left(\frac{n}{(\lceil d\rceil+1)k}\right)^{1/\lceil d\rceil} - 1\right)/(k-1) = \Omega(n^{1/\lceil d\rceil}).$$

Proof. Since G is nontrivial and twinless, $k \geq 2$ and $d > 0$. Fix a k-coloring of G. There are at least $n/(\lceil d\rceil+1)$ vertices of degree at most $\lceil d\rceil$, and at least $n/(\lceil d\rceil+1)k$ of them have the same color. The neighborhoods of these vertices are distinct, since G is twinless, so we can apply the Füredi-Tuza result to these neighborhoods where ℓ is the floor of $\lceil d\rceil(n/(\lceil d\rceil+1)k)^{1/\lceil d\rceil}/e - \lceil d\rceil$ and e is Euler's constant, since:

$$\begin{aligned}\tbinom{\lceil d\rceil+\ell}{\ell} &= \tbinom{\lceil d\rceil+\ell}{\lceil d\rceil}\\ &\leq (e(\lceil d\rceil+\ell)/\lceil d\rceil)^{\lceil d\rceil}\\ &\leq n/(\lceil d\rceil+1)k,\end{aligned}$$

We conclude that there is a set A of at least $\lceil d\rceil(n/(\lceil d\rceil+1)k)^{1/\lceil d\rceil}/e - \lceil d\rceil$ vertices whose neighborhoods have a system of strong representatives. Choose a strong representative $n(v) \in N(v)$ for each $N(v)$ with $v \in A$. These $n(v)$ can have at most $k-1$ different colors, hence there are at least

$$\left(\frac{\lceil d\rceil}{e}\left(\frac{n}{(\lceil d\rceil+1)k}\right)^{1/\lceil d\rceil} - \lceil d\rceil\right)/(k-1)$$

vertices in A all of whose assigned neighbors $n(v)$ have the same color. For these vertices, the edges $vn(v)$ form an induced matching: given two edges $un(u)$ and $vn(v)$, there cannot be edges uv or $n(u)n(v)$ by the coloring and there cannot be edges $un(v)$ or $vn(u)$ by the choice of $n(u)$ and $n(v)$. □

Remark 3.3. Consider the following bipartite graph: take a set A of ℓ vertices, and for every $d/2$-element subset of A create a new vertex and connect it to the vertices of the subset.

This graph has $n = \ell + \binom{\ell}{d/2}$ vertices, its largest induced matching has size $\ell/(d/2)$, and its average degree is $2 \cdot \frac{d}{2}\binom{\ell}{d/2} / \left(\ell + \binom{\ell}{d/2}\right) \leq d$. For d fixed, $\ell/(d/2)$ is of order $n^{2/d}$, which shows that the bound of the theorem (while not being tight) has the right form.

Remark 3.4. The preceding example can be extended to show that bounding the chromatic number is necessary: take the graph as constructed in the previous remark and add all edges between the ℓ vertices of A. Assuming $d \geq 4$, this gives a graph of average degree at most $d + 2$. However, the largest induced matching in this graph has size 1.

4. Planar Graphs and Graphs of Bounded Genus

4.1. Matchings and Induced Matchings

To find large induced matchings in graphs we can proceed in two steps: find a large matching in the graph and then turn it into an induced matching. To make this work we need to make some assumptions on the graph: to obtain a large matching, we assume an upper bound on $\alpha(G)$, the size of the largest independent set in G. To turn the matching into an induced matching, we assume that the graph is twinless and all minors of G have a large independent set.

The following result is standard [16].Although we do not apply it directly, its proof is the base of many of our later arguments.

Proposition 4.1. *Any graph G contains a matching of size at least $\frac{1}{2}(n(G) - \alpha(G))$.*

Proof. Let $M \subseteq E$ be a maximal matching in G on vertex set $V(M)$. Then $I = V - V(M)$ is an independent set. Since $|I| \leq \alpha(G)$ and $n(G) = 2|M| + |I|$, we obtain $|M| \geq \frac{1}{2}(n(G) - \alpha(G))$. □

Lemma 4.2. *Assume that any minor $H \preceq G$ of a graph G fulfills $\alpha(H) \geq c \cdot n(H)$ for some c. Then any matching M in G contains an induced matching in G of size at least $c|M|$.*

Proof. Remove all vertices not in $V(M)$ and contract the edges of M (removing duplicate edges). The resulting graph is a minor of G, and, by assumption, has an independent set of size $c|M|$. The edges in M which were contracted to the vertices in the independent set, form an induced matching in G. ☐

By this lemma a matching of size k in a planar graph contains an induced matching of size $k/4$. In [2] it is shown that every 3-connected planar graph contains a matching of size at least $(n+4)/3$, which allows us to draw the following conclusion.

Corollary 4.3. *A 3-connected planar graph on n vertices contains an induced matching of size $(n+4)/12$.*

This result is nearly tight as we will see in Remark 4.12.

To apply the two lemmas to planar graphs and graphs of bounded genus we need a generalization of Euler's theorem to hypergraphs.

Lemma 4.4. *A multihypergraph of genus at most g on n vertices has at most $2n+4g-4$ edges containing at least three vertices, unless $n=1$ and $g=0$.*

Proof. Discard all edges of size less than three and let $\mathcal{H}$ be the resulting multihypergraph. Let G be the associated bigraph embedded on a surface of genus g. It has vertex set $V(G) = V(\mathcal{H}) \cup V_E$, where $V_E = \{v_e : e \in E(\mathcal{H})\}$. We may assume that $|V_E| > 0$.

For each $v_e \in V_E$ and face f incident to v_e with $|f| \neq 3$ we add an edge drawn within f between the neighbors of v_e on the boundary of f. Repeat this step until we cannot, and let G' be the resulting graph (see Figure 1 for an illustration). While G' may have multiple edges, it will not have 2-faces.

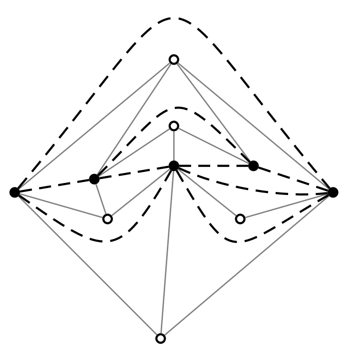

Figure 1: The graph G': vertices of $\mathcal{H}$ are black; vertices of V_E are white; edges of G are gray; edges in $E(G') - E(G)$ are dashed.

For every vertex of V_E there is a distinct face of $G' - V_E$ containing the vertex. Add edges to triangulate $G' - V_E$, and let G^* be the resulting surface triangulation, say with n^*, e^*, and f^* vertices, edges, and faces, respectively. Then $n^* = |V(\mathcal{H})| = n$, and we have observed that $|V_E| \leq f^*$. Since G^* is a triangulation, $3f^* = 2e^*$. By Euler's formula we get $2 - 2g = n^* - e^* + f^* = n - \frac{1}{2}f^*$, so $|E(\mathcal{H})| = |V_E| \leq f^* = 2n + 4g - 4$, as desired. □

If $\mathcal{H}$ is a hypergraph of genus g such that all edges have size 2, we can take the associated bigraph G of genus g and contract away all the vertices that correspond to edges of $\mathcal{H}$. This produces a graph of genus g with $|V(\mathcal{H})|$ vertices and $|E(\mathcal{H})|$ edges, to which we may apply the following consequence of Euler's Theorem.

Lemma 4.5 (Euler)**.** *A graph of genus g on n vertices contains at most $3n + 6g - 6$ edges if $n \geq 3$.*

By partitioning the edge set of a hypergraph into sets of edges of size at least three, edges of size two, and edges that contain a single vertex, we can derive the following.

Lemma 4.6. *A hypergraph of genus at most g on n vertices has at most $6n + 10g - 10$ edges if $n \geq 3$.*

Finally, we need results on coloring graphs on surfaces of bounded genus. There are sharp results: Heawood's theorem for genus $g \geq 1$, and the Four-Color Theorem for $g = 0$.

Lemma 4.7 (Heawood [16])**.** *A graph of genus at most $g \geq 1$ can be colored using at most $(7 + \sqrt{1 + 48g})/2$ colors.*

The bound remains valid for $g = 0$ by the Four-Color Theorem. We are now ready to give a lower bound on the size of induced matchings in twinless graphs of bounded genus. This includes the planar case, for which we will improve the bound in the next section.

Theorem 4.8. *A nontrivial twinless graph of genus at most g on n vertices contains an induced matching of size at least $(n - 10g + 10)/[6.5(7 + \sqrt{1 + 48g})]$.*

Proof. Let G be as in the statement of the theorem. If $n \leq 4$, then $g = 0$ so an induced matching of size 1 suffices; since G is nontrivial and twinless, $E(G) \neq \emptyset$ and this exists. Thus, we assume that $n > 4$. If G has a component H with $n(H) = 3$, then we remove it and apply induction; together with one edge in H, this yields a sufficiently large induced matching. Thus, we assume that there is no such component.

Let $M \subseteq E(G)$ be a maximal matching in G on vertex set $V(M)$, chosen to be incident to the maximum number of vertices of degree 2. Then $I = V(G) - V(M)$ is an independent set. If $N(x) = \{u, v\}$ with $uv \in M$ and $x \in I$, then, by the choice of M, both u and v must have degree 2. But then $\{x, u, v\}$ would induce a 3-vertex component, a possibility we already excluded. Hence for each edge $uv \in M$, no vertex $x \in I$ has $N(x) = \{u, v\}$.

Let $\mathcal{H}$ be the hypergraph with vertex set $V(M)$ and edge set $\{N(x) : x \in I\} \cup \{\{u, v\} : uv \in M\}$. $\mathcal{H}$ has no multiple edges because G is twinless, and since no vertex $x \in I$ has $N(x) = \{u, v\}$. Clearly $\mathcal{H}$ has the same genus as G, which is at most g.

If $|V(M)| \geq 3$, then Lemma 4.6 applies, and $\mathcal{H}$ has at most $6|V(M)| + 10g - 10$ edges. Thus $|I| + |M| \leq 12|M| + 10g - 10$, and since $n = |I| + 2|M|$, G has a matching of size at least $(n - 10g + 10)/13$. Otherwise, $|M| \leq 1$. Then G has at most one component with edges, which must be K_3 or a star. Since G is twinless, a nontrivial star must be K_2. Also, G has at most one isolated vertex, so $n \leq 4$, which is a contradiction.

By Heawood's theorem (Lemma 4.7) and the Four-Color Theorem, a graph of genus at most g can be colored using at most $(7 + \sqrt{1 + 48g})/2$ colors. Hence, G and any of its minors always contain independent sets on a $2/(7 + \sqrt{1 + 48g})$-fraction of their vertices. Then by Lemma 4.2, G has an induced matching of size at least $2(n - 10g + 10)/[13(7 + \sqrt{1 + 48g})] = (n - 10g + 10)/[6.5(7 + \sqrt{1 + 48g})]$. □

In particular, a planar twinless graph always contains an induced matching of size $(n + 10)/52$. As we mentioned, we will improve this bound for planar graphs in Section 4.2. Here we present a simple consequence not involving the concept of twinlessness:

Corollary 4.9. *A planar graph of minimum degree at least* 3 *on* n *vertices contains an induced matching of size at least* $(n + 4)/24$.

Proof. Let M be a maximal matching, let $I = V(G) - V(M)$, and let $\mathcal{H}$ be the multihypergraph with vertex set $V(M)$ and edge set $\{N(v) : v \in I\}$. By Lemma 4.4, $\mathcal{H}$ has at most $2|V(M)| - 4$ edges, so $|I| \leq 2|V(M)| - 4$. Then $n \leq 3|V(M)| - 4$, so $|M| = |V(M)|/2 \geq (n + 4)/6$. Using Lemma 4.2, the matching contains an induced matching of size at least $(n + 4)/24$. □

The condition in Lemma 4.2 can be replaced by an average degree condition if we are looking at graph classes that are not closed under minors.

Lemma 4.10. *Assume that G and each of its subgraphs has average degree at most d. Then any matching M in G contains an induced matching in G of size at least $|M|/(2d-1)$.*

Proof. Let $G_M = G[V(M)]$ be the graph G restricted to vertices in $V(M)$. An induced matching in G_M will be an induced matching in G. Let $d(v)$ denote the degree of v in G_M. By assumption, the average degree of G_M is at most d. Since

$$\sum_{uv \in M} (d(u) + d(v)) = \sum_{v \in V(M)} d(v) \leq d|V(M)|,$$

there is an edge $uv \in M$ such that $d(u) + d(v) \leq 2d$. Removing the two vertices and their neighbors destroys at most $1 + (d(u) - 1) + (d(v) - 1) \leq 2d - 1$ edges of the matching M in G_M. Thus the resulting graph contains a bipartite balanced graph with a perfect matching M' of size at least $|M| - (2d-1)$ in M. We recurse on $G_{M'}$. □

4.2. An Improved Bound For Planar Graphs

In this section we improve the bound on induced matchings in planar graphs given in Theorem 4.8.

Theorem 4.11. *A nontrivial twinless planar graph G contains an induced matching of size at least $n(G)/40$.*

Proof. Let M be a maximal matching of G; then $I = V(G) - V(M)$ is an independent set. We write $n = n(G)$; c is a constant to be determined later. Let I_0 be the set of isolated vertices in I; by assumption $|I_0| \leq 1$. If I has at least $4n/c$ vertices of degree 1, let I_1 be the set of such vertices. Since G is twinless, no two vertices in I_1 share the same neighbor, and $|N(I_1)| = |I_1|$. By the Four-Color Theorem, at least n/c vertices in $N(I_1)$ form an independent set in G. Now the edges joining these vertices to their neighbors in I_1 form an induced matching in G of size at least n/c.

A similar argument can be used to bound the number of vertices of degree 2 in I in terms of the size of the induced matching. Let I_2 be the set of vertices in I of degree 2. Let G_2 be the graph formed by taking the induced graph on $N(I_2)$, and for each vertex $w \in I_2$, if w is adjacent to vertices w_1, w_2 with $w_1w_2 \notin E(G)$, then we add the edge w_1w_2 to G_2. Then $n(I_2) \leq e(G_2)$. Since w has degree 2 and G is planar, each new edge w_1w_2 can be drawn near the edges w_1w, ww_2 in a planar drawing of G. Hence G_2 is planar, and $e(G_2) \leq 3n(G_2)$. By the Four-Color Theorem, G_2 has an independent set of size at least $n(G_2)/4$. By picking a neighbor in I_2 of every vertex in this independent set we obtain an induced matching in G of at least $n(G_2)/4 \geq n(I_2)/12$ vertices. It follows from this that if I contains at least $12n/c$ vertices of degree 2, then G has an induced matching of at least n/c edges.

G has at least one edge since G is nontrivial and twinless, so $|M| \geq 1$ and $V(M) \geq 2$. Then we can apply Lemma 4.4 to $\mathcal{H}$, so the number of vertices in I of degree at least 3 is bounded by $2|V(M)|-4$.Therefore, assuming that there is no induced matching of at least n/c edges whose edges are all incident to vertices in I, we have $|I|-1-16n/c \leq |I|-|I_0|-|I_1|-|I_2| \leq 2|V(M)|-4$. Since $|I|+|V(M)| = n$, we can conclude that $|V(M)| \geq n(c-16)/(3c)$. If $V(M)$ contains at least $8n/c$ vertices, then by Lemma 4.2, G has an induced matching of at least n/c edges. By choosing $c = 40$ so that $8n/c = n(c-16)/(3c)$, we can conclude that G has an induced matching of at least $n/40$ edges. □

Remark 4.12. We do not have a matching upper bound to complement Theorem 4.11, but we can get close. The following construction builds a planar graph whose largest induced matching has size $(n+10)/27$.

We first build a basic gadget for the construction. Draw a K_4 on vertex set V_4. Add a degree 3-vertex to each face. Add a degree 1 vertex attached to each vertex of V_4. Add a degree 2 vertex adjacent to each pair of vertices in V_4 (drawn near an edge of the original K_4). Now exactly two vertices of V_4 will be on the outer face. Note that the gadget has 18 vertices; if we remove all vertices of degree 1 and 2 it has 8 vertices.

For convenience, we describe the full construction by first drawing a framework for the graph, before using it to construct the desired graph. Draw a $2k$-cycle on vertices $v_1, \ldots, v_{2k}$. On the interior of the cycle add edges v_1v_j for $3 \leq j \leq 2k-1$, and on the exterior of the cycle add edges $v_{2k}v_j$ for $2 \leq j \leq 2k-2$. Note that there are no multiple edges, and that the faces are incident to distinct 3-sets of vertices. Now we construct the desired graph: Add a vertex of degree 3 to each face. For $1 \leq j \leq k$ replace the edge $v_{2j-1}v_{2j}$ by a gadget with v_{2j-1} and v_{2j} as its exposed vertices, and subdivide every other edge of the framework.

By the construction, we obtain a planar twinless graph. The framework is a triangulation on $2k$ vertices, $6k-6$ edges, and $4k-4$ faces, so our final graph has $18k + (5k-6) + (4k-4) = 27k - 10$ vertices.

Note that any edge in the graph has at least one endpoint in V_4 of some gadget, and that the neighborhood of that endpoint contains all of V_4 from that gadget, and that the gadget minus V_4 is an independent set. Therefore an induced matching contains at most one edge incident to that gadget. Thus the maximum size of an induced matching is bounded above by the number of gadgets, k, and obviously it equals k. In terms of the total number of vertices $n = 27k - 10$, this is $(n + 10)/27$.

By deleting the vertices of degree 1 and 2, we get a twinless planar graph of minimum degree 3 on $8k + (4k - 4) = 12k - 4$ vertices, and a maximum induced matching of size k. In terms of the total number of vertices n, this is $(n + 4)/12$. For comparison the bound from Corollary 4.9 is $(n + 4)/24$. We can further modify this example to show that the bound given in Corollary 4.3 is nearly tight: For each gadget, take its degree 3 vertex x incident to its outer face, which lies in a face f of the framework, and identify x with the degree 3 vertex added to f. The resulting graph is a twinless, 3-connected planar graph on $(12k - 4) - k = 11k - 4$ vertices. In terms of the total number of vertices n, this is $(n + 4)/11$.

5. Induced Matchings in Outerplanar Graphs

Our goal in this section is to show that every connected outerplanar graph G with minimum degree 2 has an induced matching of size $\lceil \frac{n}{7} \rceil$. Since outerplanar graphs have minimum degree at most 2 the result applies to all outerplanar graphs that do not have isolated vertices or leaves.

The result is sharp: Figure 2 shows an example of a graph in which the size of the maximum induced matching is exactly $\lceil n/7 \rceil$. A graph in this family consists of a cycle of length 2ℓ ($\ell \geq 3$) with ℓ gadgets attached as indicated in the figure. The total number of vertices in this graph is 7ℓ, and it is easy to verify that a maximum induced matching has size exactly ℓ.

Before deriving the sharp bound, we show how to apply the approach from the previous section to obtain easy lower bounds (Corollary 5.2) for outerplanar graphs of minimum degree 2 and 2-connected outerplanar graphs.

Lemma 5.1. *Let G be an outerplanar graph of minimum degree* 2 *and let $n = n(G)$. Then G contains a matching of size at least $\lceil n/3 \rceil$. If G is* 2*-connected, then it has a matching of size at least $\lfloor n/2 \rfloor$.*

Proof. If G is a 2-connected outerplanar graph, then it is Hamiltonian (see [16]), and G has a matching of size $\lfloor \frac{n}{2} \rfloor \geq \lceil \frac{n}{3} \rceil$. Otherwise, let B be a leaf block in the block decomposition of G, and let u be the cut-vertex of G in B. If u is not a leaf in $G - (V(B) - u)$, then let $H = B - u$, and $G' = G - V(H)$. If u is a leaf in $G - (V(B) - u)$, let P be the minimal path in $G - (V(B) - u)$ from u to another vertex v of degree not equal to 2 (actually at least 3, since G has minimum degree 2), and let $H = B \cup (P - v)$, and

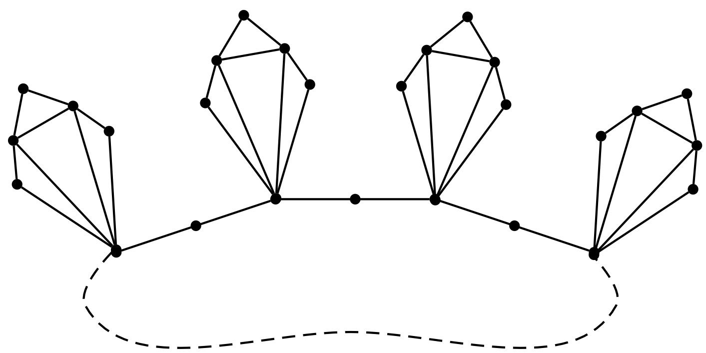

Figure 2: An illustration of a family of outerplanar graphs for which the lower bound on the size of an induced matching is tight.

$G' = G - V(H)$. In both cases, H has a Hamiltonian path and $n(H) \geq 2$ because B is Hamiltonian, and G' has minimum degree 2 (and is non-empty). Applying induction to G' and using the Hamiltonian path to get a matching of H, we get a matching in G of size $\lceil n(G')/3 \rceil + \lfloor n(H)/2 \rfloor$, which is at least $\lceil n/3 \rceil$. □

Note that the bound $\lceil \frac{n}{3} \rceil$ is asymptotically tight for matchings in outerplanar graphs, as illustrated in Figure 3.

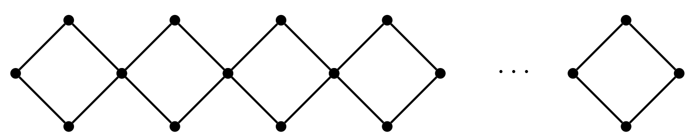

Figure 3: $3\ell + 1$ vertices with a maximum matching of size $\ell + 1$.

Outerplanar graphs are 3-colorable ([16, Exercise 6.3.3]) and closed under taking minors, so we can apply Lemma 4.2 with $c = 1/3$. Together with Lemma 5.1, we get the following easy lower bounds.

Corollary 5.2. *Let G be an outerplanar graph of minimum degree 2 and let $n = n(G)$. Then $\mathrm{mim}(G) \geq \lceil n/9 \rceil$. If G is 2-connected, then $\mathrm{mim}(G) \geq \lceil \lfloor n/2 \rfloor / 3 \rceil$.*

To derive the tight bound $\lceil \frac{n}{7} \rceil$ for outerplanar graphs of maximum degree 2, we first consider a special case, which will also arise later in the proof of the main result.

Lemma 5.3. *Suppose that G is a connected graph for which the block-cutpoint tree is a path and each block is a triangle or a cut-edge. Then* $\mathrm{mim}(G) \geq \lceil \frac{n(G)-1}{4} \rceil$.

Proof. If G has exactly one block, $n(G) \geq 3$ and $\mathrm{mim}(G) = 1$, which suffices.

If a leaf block B is a triangle, then we can apply induction to $G - V(B)$ to obtain an induced matching in $G - V(B)$ of size at least $\lceil \frac{n(G)-3-1}{4} \rceil$. To this we add the edge of B that is not incident to the cut-vertex of G in B. This gives us an induced matching of G of size at least $\lceil \frac{n(G)}{4} \rceil$.

If a leaf block of G consists of an edge uv with $\deg(v) = 1$, let B be the other block incident to u in the block-cutpoint tree. We can apply induction to $G - V(B) - v$ if $n(G) - n(B) - 1 \geq 2$, which gives us an induced matching of size at least $\lceil \frac{n(G)-4-1}{4} \rceil$; if $n(G) - n(B) - 1 < 2$ then an empty matching has size $\lceil \frac{n(G)-4-1}{4} \rceil$. We can add the edge uv to obtain an induced matching of at least $\lceil \frac{n(G)-1}{4} \rceil$. □

Corollary 5.4. *Let G be a 2-connected outerplanar graph with exactly one non-leaf face, such that every leaf face is a 3-face. Then for any vertex v,* $\mathrm{mim}(G - v) \geq \lceil \frac{n(G)}{6} \rceil$.

Proof. If v is incident to the non-leaf face, apply the previous lemma to $G-v$; this suffices since $\lceil \frac{n(G)-2}{4} \rceil \geq \lceil \frac{n(G)}{6} \rceil$. So, assume not. Then v is incident to only one face, a leaf face, and v has degree 2. Clearly $\mathrm{mim}(G - v) \geq 1$, so we are done if $3 \leq n(G) \leq 6$.

Suppose that $n(G) \geq 8$. Let u be a neighbor of v and apply Lemma 5.3 to $G - \{u, v\}$. We get $\mathrm{mim}(G - \{u, v\}) \geq \lceil \frac{n(G)-2-1}{4} \rceil \geq \lceil \frac{n(G)}{6} \rceil$.

Suppose that $n(G) = 7$, and note that $\lceil \frac{n(G)}{6} \rceil = 2$. Let the boundary of $G-v$ be the cycle $(u_1, \ldots, u_6, u_1)$ with $N(v) = \{u_1, u_6\}$. By the restrictions on faces, any chord of that cycle must be of the form $u_i u_{i+2}$ for some $1 \leq i \leq 4$. Then $\{u_1u_2, u_4u_5\}$ is an induced matching unless u_2u_4 is an edge, and $\{u_2u_3, u_5u_6\}$ is an induced matching unless u_3u_5 is an edge. Those two edges would cross, so $\mathrm{mim}(G - v) \geq 2$. □

To prove the main result of this section, that a connected outerplanar graph G of minimum degree 2 has an induced matching of size $\lceil \frac{n}{7} \rceil$, we use induction on graph components created when we remove certain vertices from the graph. We need to ensure that each of these components has minimum degree 2 so that the inductive hypothesis applies. Since this may not be true after the removal of a cut-set from the graph, we introduce a *patching operation* that patches a component so that its minimum degree is 2.

Definition 5.5. Let H be a connected outerplanar graph with $n(H) \geq 4$ and at most two degree 1 vertices. The result of the *patching operation* on H is the graph H' defined as follows.

(a) If there is no degree 1 vertex in H let $H' = H$.

(b) If there is exactly one degree 1 vertex v in H, let v' be its neighbor. If $\deg_H(v') \geq 3$, let $H' = H - v$. Since $n(H) > 2$, otherwise $\deg_H(v') = 2$. Then let w be the other neighbor of v', let w' be a vertex after w on the boundary walk in $H - \{v, v'\}$, and let $H' = (H - v) + v'w'$.

(c) If there are two degree 1 vertices u and v in H that are adjacent to the same vertex, then $H - u$ is a connected outerplanar graph with exactly one degree 1 vertex and $n(H - u) > 2$. Define H' by applying the previous patching operation to $H - u$.

(d) Otherwise, the degree 1 vertices u, v and their neighbors u', v' (respectively) are all distinct. Remove u from H and add the edge $u'v$. Let H' be the resulting graph.

Proposition 5.6. *Let H be a connected outerplanar graph with $n(H) \geq 4$ and at most two degree 1 vertices. Moreover, assume that if H has two degree 1 vertices u and v, then adding a path between u to v leaves H outerplanar. Let H' be the result of applying the patching operation to H.*

Then H' is an outerplanar graph with minimum degree 2, and $\mathrm{mim}(H) \geq \mathrm{mim}(H')$. *Also $n(H') \geq n(H) - 2$, and $n(H') \geq n(H) - 1$ except in case (c).*

Proof. Given an outerplane embedding of H, it is easy to get an outerplane embedding of H': in cases (b) and (c) if $v'w'$ is added, draw it near v', w, w' in the outer face of H; in case (d) draw a u, v-path P such that $H \cup P$ is drawn as an outerplane embedding, then contract $(P - v) \cup u'u$ to u'. Hence, H' is outerplanar.

From the definition of the patching operation, it is clear that H' has minimum degree 2. Note that $n(H') = n(H)$ in case (a), $n(H') = n(H) - 1$ in cases (b) and (d), and $n(H') = n(H) - 2$ in case (c).

To show that $\mathrm{mim}(H) \geq \mathrm{mim}(H')$, let M' be a maximum induced matching in H'. M' is an induced matching in H except in cases (b) and (c) when $v'w' \in M'$ and in case (d) when $u'v \in M'$. Replacing $v'w'$ by $v'v$, or $u'v$ by $u'u$, gives an induced matching in H. Therefore H has an induced matching of size $\mathrm{mim}(H')$, which completes the proof. □

Theorem 5.7. *A connected outerplanar graph G of minimum degree 2 has an induced matching of size $\lceil \frac{n(G)}{7} \rceil$.*

Proof. The proof is by induction. Let $n = n(G)$. Clearly the statement is true if $n \leq 7$. Assume that $n \geq 8$ and that the statement is true for any graph with fewer than n vertices.

If G is 2-connected, then $\text{mim}(G) \geq \lceil \lfloor n/2 \rfloor /3 \rceil$ by Corollary 5.2 and this is at least $\lceil n/7 \rceil$. Thus we may assume that G contains a cut-vertex; pick a cut-vertex u in G which is in at most one non-leaf block. Let $B_1, \ldots, B_\ell$ be all the leaf blocks containing u, let $B_0 = G - \bigcup_{i=1}^{\ell}(V(B_i) - u)$, and let $n_i = n(B_i)$, for $i = 0, \ldots, \ell$. Note that $n_0 + n_1 + \cdots + n_\ell = n + \ell$.

We first show that, for all $1 \leq i \leq \ell$, $B_i - u$ contains an induced matching of size $\min\{\lceil \frac{n_i}{6} \rceil, \lceil \frac{n_i+4}{7} \rceil\}$. For this purpose, let B_i' be the graph obtained from B_i by deleting the chord of each leaf face of length 3, for $1 \leq i \leq \ell$. Note that any leaf face in B_i' must have length at least 4.

The case that B_i' is a cycle is easy, since we can apply Corollary 5.4, so we focus on the other case.

Claim 1. For each i with $1 \leq i \leq \ell$, if $n_i \geq 7$ and B_i' is not a cycle, we can assume that there are exactly two leaf faces in B_i', each has length 4 or 5, and both are incident to u.

Proof. Suppose that B_i' has a leaf face with boundary $F = (u_1, \ldots, u_r, u_1)$ so that u_1u_r is a chord and $u \notin U := \{u_j : j \leq \min\{r, 5\}\}$. Let $H = G - U$. Then H is an outerplanar graph with at most two degree 1 vertices, and if there are two degree 1 vertices, then there is a path between them in G which leaves the drawing outerplanar. Since $B_i' - \{u_2, \ldots, u_{r-1}\}$ has a face of length at least 4, $n(B_i' - U) \geq 2$. Since G has at least two blocks, $n(G - V(B_i')) \geq 2$. Therefore $n(H) \geq 4$, so we may apply the patching operation to H to obtain a graph H'. H' is a connected outerplanar graph with minimum degree two. By induction, $\text{mim}(H') \geq \lceil \frac{n(H')}{7} \rceil$. Since $n(H') \geq n(H) - 2$ and $\text{mim}(H) \geq \text{mim}(H')$ by Proposition 5.6, we have $\text{mim}(H) \geq \lceil \frac{n(H)-2}{7} \rceil \geq \lceil \frac{n(G)-7}{7} \rceil$.

For any edge of B_i incident to u_2u_3, the other endpoint must be $u_j \in V(B_i')$ with $1 \leq j \leq r$ by the choice of F, with $j \leq 5$ because of the construction of B_i' from B_i; hence $u_j \in U$. Therefore, a maximum induced matching in H plus the edge u_2u_3 is an induced matching in G. We conclude that $\text{mim}(G) \geq \lceil \frac{n(G)-7}{7} \rceil + 1 = \lceil \frac{n(G)}{7} \rceil$, as desired. So we may assume that B_i' has no such leaf face.

Now suppose that there is a leaf face of B_i' with boundary $F = (u_1, \ldots, u_r, u_1)$ so that u_1u_r is a chord, and $u = u_j$ for some j with

$2 \le j \le r-1$. Since B_i' is not a cycle, the weak dual is a nontrivial tree, and hence B_i' has at least two leaf faces. However, any other leaf face must be the first kind we considered, a contradiction.

It follows that any leaf face of B_i' must be incident to a chord which is incident to u, and that it must have length at most 5. There can be at most two such leaf faces, and since the weak dual of B_i' is a nontrivial tree, there are exactly two. $\square$

Claim 2. For each i with $1 \le i \le \ell$, if $n_i \ge 7$ and B_i' is not a cycle, then $B_i - u$ contains an induced matching M_i of size at least $\min\{\lceil \frac{n_i}{6} \rceil, \lceil \frac{n_i+4}{7} \rceil\}$.

Proof. By Claim 1, the leaf faces of B_i' have boundaries $F = (u_1, \dots, u_r, u_1)$ and $F' = (u_1', \dots, u_s', u_1')$ where $u_1 u_r$ and $u_1' u_s'$ are chords, $u_1 = u_1' = u$, and $4 \le r, s \le 5$. (Note that $u_r = u_s'$ is possible.) Let H be the graph obtained from B_i by removing the vertices in $F \cup F'$. Note that $n_i \le n(H) + 9$.

If $n(H) \le 3$, then the edges $u_2 u_3$ and $u_2' u_3'$ give an induced matching in $B_i - u$ of size $2 \ge \lceil \frac{n(H)+9}{6} \rceil \ge \lceil \frac{n_i}{6} \rceil$.

Suppose that $n(H) \ge 4$. Since H contains a Hamiltonian path (since B_i has a Hamiltonian cycle including $u_r, \dots, u_2, u, u_2', \dots, u_s'$), it has at most two vertices of degree 1. Therefore, we can apply the patching operation to H to obtain H'. Case (c) is not used since H has a Hamiltonian path and $n(H) \ge 4$, so $n(H) \ge n(H') - 1$. Inductively, $\mathrm{mim}(H') \ge \lceil \frac{n(H')}{7} \rceil$, so $\mathrm{mim}(H) \ge \lceil \frac{n(H)-1}{7} \rceil$. Now any induced matching in H plus edges $u_2 u_3$ and $u_2' u_3'$ gives an induced matching M_i in $B_i - u$. It follows that $\mathrm{mim}(B_i - u) \ge 2 + \mathrm{mim}(H) \ge 2 + \lceil \frac{n(H)-1}{7} \rceil \ge \lceil \frac{n_i+4}{7} \rceil$. $\square$

Consider any i with $1 \le i \le \ell$. If $n_i \le 6$, then $\lceil \frac{n_i}{6} \rceil = 1$, and $\mathrm{mim}(B_i - u) \ge 1$ since B_i is 2-connected. If $n_i \ge 7$ and B_i' is a cycle, then by Corollary 5.4, $B_i - u$ contains an induced matching M_i of size at least $\lceil \frac{n_i}{6} \rceil$.

With Claim 2, we can now assume that $B_i - u$ contains an induced matching M_i of size at least $\min\{\lceil \frac{n_i}{6} \rceil, \lceil \frac{n_i+4}{7} \rceil\}$ for all $1 \le i \le \ell$.

Let $M = \bigcup_{i=1}^{\ell} M_i$. Let $H = B_0 - u$ and note that H has at most two degree 1 vertices. If $n(H) \ge 4$, apply the patching operation to H to obtain an outerplanar graph H' of minimum degree 2. Applying the inductive statement to H' we conclude that $\mathrm{mim}(H) \ge \lceil \frac{n_0-1-2}{7} \rceil$. If $n(H) \le 3$ then $\lceil \frac{n_0-3}{7} \rceil = 0$, so in any case $B_0 - u$ contains an induced matching M_0 of size at least $\lceil \frac{n_0-3}{7} \rceil$. Since no edge in $M \cup M_0$ is incident to u, $M \cup M_0$ is an induced matching in G.

For the following calculations, recall that $n_0+n_1+\cdots+n_\ell = n+\ell$. Note that $\min\{\lceil \frac{n_i}{6}\rceil, \lceil \frac{n_i+4}{7}\rceil\} \geq \lceil \frac{n_i}{7}\rceil$. If $|M_i| \geq \frac{n_i+2}{7}$ for some i with $1 \leq i \leq \ell$, then:

$$|M \cup M_0| \geq \sum_{j=1, j\neq i}^{\ell} \lceil \frac{n_j}{7}\rceil + \frac{n_i+2}{7} + \lceil \frac{n_0-3}{7}\rceil \geq \lceil \frac{n-1+\ell}{7}\rceil \geq \lceil \frac{n}{7}\rceil.$$

Thus we may assume that $\lceil \frac{n_i}{6}\rceil \leq |M_i| \leq \frac{n_i+1}{7}$ for all $1 \leq i \leq \ell$; it follows that each $n_i = 6$ and $|M_i| = 1$. Then $|M \cup M_0| \geq \ell + \lceil \frac{n_0-3}{7}\rceil \geq \lceil \frac{n_0}{7}\rceil$. □

6. Applications to Parameterized Computation

In this section we apply our previous results to obtain parameterized algorithms for IM on graphs of bounded genus. Let (G, k) be an instance of IM where G has n vertices and genus g for some integer constant $g \geq 0$.

6.1. A Problem Kernel

We first show how to kernelize the instance (G, k) when G is planar (i.e., for the case $g = 0$). We then extend the results to graphs with genus g for any integer constant $g > 0$.

Theorem 4.11 shows that any nontrivial twinless planar graph on n vertices has an induced matching of at least $n/40$ edges. Observing that if u is a vertex in G that has a twin then $\mathrm{mim}(G) = \mathrm{mim}(G-u)$, by repeatedly removing every vertex in G with a twin, we end up with a twinless graph G' such that G has an induced matching of size k if and only if G' does. If $k \leq n(G')/40$ then the instance (G', k) of IM can be accepted; otherwise, the instance (G', k) is a kernel of (G, k) with $n(G') \leq 40k$, and we can work on (G', k).

Therefore, our task amounts to reducing the graph G to the twinless graph G'. We describe next how this can be done in linear time.

Assume that G is given by its adjacency list and that the vertices in G are labeled by the integers $1, \ldots, n$. We can further assume that the neighbors of every vertex appear in the adjacency list in increasing order. If this is not the case, we create the desired adjacency list by enumerating the vertices in increasing order, and inserting each vertex in the neighborhood list of each of its adjacent vertices. This can easily be done in $O(n)$ time.

For every vertex v of degree d, we associate a d-digit number $x_v = v_1 \cdots v_d$, where $v_1, \ldots, v_d$ are the neighbors of v in the order they appear in the adjacency list of v (i.e., in increasing order). We perform a radix sort on the numbers associated with the vertices of G using only the first three or less (leftmost) digits of these numbers. Since each digit is a number in the range $1 \dots n$, and there are at most $O(n)$ numbers (twice the number of edges in the planar graph), radix sort takes $O(n)$ time. Let π be this sorted list. Observe that two vertices u and v are twins if and only if $x_u = x_v$. Moreover, since the graph is planar, and hence does not contain the complete bipartite graph $K_{r,r}$ for any integer $r \geq 3$, any twin vertices of degree at least 3 must have their numbers adjacent in π (otherwise there will be at least 3 vertices with the same neighborhood). Therefore, we can recognize the twins in G as follows. Process the numbers in π in order: Let x_u and x_v be two adjacent numbers in π, and assume that x_u appears before x_v. We check whether u and v are twins by comparing the corresponding digits of x_u and x_v. If u and v are twins, we mark u. When we have finished this process, we remove all marked vertices from the graph. We let G' be the resulting graph. Since for each number x_u in π we spend time proportional to the number of digits in x_u and that of the number appearing next to x_u in π, the running time is proportional to the sum of the degrees of the vertices in G, which is $O(n)$. We have the following theorem.

Theorem 6.1. *Let (G, k) be an instance of* IM *where G is a planar graph on n vertices. Then in $O(n)$ time we can compute an instance (G', k') where (G', k') is a kernel of (G, k) and such that either $n(G') \geq 40k'$ and we can accept the instance (G, k), or $n(G') < 40k'$.*

The above theorem gives a kernel of size $40k$ for Planar-IM, and is a significant improvement on the results in [11] where a kernel of size $O(k)$ was derived without the constant in the asymptotic notation being specified. The above results give a concrete value for the bound on the kernel size. Moreover, this value is moderately small and the analysis techniques are much simpler when compared to the technique of decomposing a planar graph into regions used in [11].

Removing twin vertices in graphs of genus g is similar to the planar case. Using Euler's formula on $K_{r,r}$ with the fact that faces in an embedded bipartite graph have length at least 4, the following result can be shown easily.

Proposition 6.2. *A graph with genus g does not contain the complete bipartite graph $K_{r,r}$ for any $r > 2 + 2\sqrt{g}$.*

Using Theorem 4.8 and Proposition 6.2, Theorem 6.1 can now be generalized to graphs with bounded genus.

Theorem 6.3. *Let (G, k) be an instance of* IM *where G is a graph on n vertices with genus g. Then in $O(gn)$ time we can compute an instance (G', k') where (G', k') is a kernel of (G, k) and such that either $n(G') \geq (7 + \sqrt{1 + 48g})6.5k'/ + 10g - 10$ and we can accept the instance (G, k), or $n(G') < (7 + \sqrt{1 + 48g})6.5k' + 10g - 10$.*

6.2. Parameterized Algorithms for IM *on Graphs with Bounded Genus*

We again treat the planar case first. Assume that we have an instance (G, k) of Planar-IM. By Theorem 6.1, we can assume that after an $O(n)$ preprocessing time, the number of vertices n in G satisfies $n \leq 40k$. We will show how to design a parameterized algorithm for the Planar-IM problem. Our algorithm is a bounded-search-tree algorithm that uses the Lipton-Tarjan separator theorem [10]. Our results answer an open question posed by [11] of whether a bounded-search-tree algorithm exists for Planar-IM. We also show at the end of this section how these results can be extended to bounded genus graphs.

Theorem 6.4 (Lipton, Tarjan [10]). *Given a planar graph $G = (V, E)$ on n vertices, there is a linear time algorithm that partitions V into vertex-sets A, B, S such that:*

1. $|A|, |B| \leq 2n/3$*;*
2. $|S| \leq \sqrt{8n}$*; and*
3. *S separates A and B, i.e. there is no edge between a vertex in A and a vertex in B.*

Given an instance (G, k) of Planar-IM, where $G = (V, E)$ and $|V| = n$, we partition V into vertex-sets A, B, S according to the Lipton-Tarjan theorem. Let G_A, G_B, and G_S be the subgraphs of G induced by the vertices in A, B, and S, respectively. The idea is simple: separate the graph by enumerating all possible types for the vertices in S, and then use a divide-and-conquer approach. However, special care needs to be taken when enumerating the vertices in S as this enumeration is not straightforward. We outline the general approach below.

Each vertex u in S is either an endpoint of an edge in the induced matching or not. Therefore, we assign each vertex u one of two possible types: type 1 if u is an endpoint of an edge in the induced matching and 0 if it is not (type 1 has subtypes as we will see presently). Suppose that we have assigned a type to every vertex u in S. If u is of type 0, we simply remove u (and its incident edges) from G. If u is of type 1 and there is an edge uu' with $u' \in S$ and u' is of type 1, then uu' has to be an edge in the induced matching if our enumeration is correct. Therefore, we can add uu' to the matching and remove all the neighbors of u and u' from G. If u is of type 1, and there is no vertex $u' \in S$ of type 1 so that uu' is an edge, then we refine the type of u: we assign it type 1_A to denote that u is matched to a vertex in A and type 1_B to denote that it is matched to a vertex in B. In the former case, we add u to G_A and remove all its neighbors in G_B, and in the latter case we add u to B and remove all its neighbors in G_A.

After assigning each vertex in S a type in $\{0, 1_A, 1_B\}$, and updating the graph according to the above description, G_A and G_B are separated, and we can recurse on them to compute an induced matching M_A of G_A and M_B of G_B. Let M be $M_A \cup M_B$ plus all the edges uu' for which $u, u' \in S$ and both were of type 1. The enumeration can choose poorly, and M might not be an induced matching, so we need to verify that it is, before returning it.

If there exists an induced matching of at least k edges in G, then it is not difficult to see that at least one enumeration will return such an induced matching. Otherwise, no enumeration can find an induced matching of at least k edges, and we can reject the instance.

Finally, note that in the recursive calls, some of the vertices in G_A and G_B may have already been assigned type 1, and we need to respect the assigned types in any possible future enumeration of those vertices in G_A and G_B.

The running time of the algorithm can be expressed using the following recurrence relation:

$$T(n) \leq \begin{cases} O(1) & \text{if } n = O(1) \\ 2 \cdot 3^{\sqrt{8n}} T(2n/3 + \sqrt{8n}) + O(n) & \text{otherwise.} \end{cases}$$

By solving the above recurrence relation, we get $T(n) = O(2^{25\sqrt{n}})$. Noting that $n \leq 40k$, we have the following theorem:

Theorem 6.5. *In time $O(2^{159\sqrt{k}}+n)$, it can be determined whether a planar graph on n vertices has an induced matching of at least k edges.*

The above results can be extended to bounded genus graphs. Let G be a nontrivial twinless graph on n vertices with genus g. By Theorem 4.8, G has an induced matching of size at least $(n-10g+10)/(6.5(7+1\sqrt{1+48g}))$. Therefore, we can assume that $n < (7+\sqrt{1+48g})6.5k+10g-10$; otherwise, we can accept the instance (G, k) of the induced matching problem. The following theorem by Djidjev and Venkatesan is the analogue of the Lipton-Tarjan theorem for bounded genus graphs:

Theorem 6.6 (Djidjev, Venkatesan [3]). *Let G be a graph on n vertices and genus g. There is a linear time algorithm that partitions the vertices of G into three sets A, B, C, such that no edge joins a vertex in A with a vertex in B, $|A|, |B| \leq n/2$, and $|C| \leq c_0\sqrt{(g+1)n}$, where c_0 is a fixed constant.*

Using the above theorem, and the same approach used for Planar-IM, we conclude with the following theorem:

Theorem 6.7. *Let G be a graph on n vertices with genus g. In time $O(2^{O(\sqrt{gk})}+n)$ for $g \geq 1$, and $O(2^{O(\sqrt{k})}+n)$ for $g = 0$, it can be determined whether G has an induced matching of at least k edges.*

Due to the large constant in the exponent of the running time of the above algorithms, it is clear that these algorithms are far from being practical. We shall present in the next section more practical parameterized algorithms for IM on bounded genus graphs.

7. Practical Algorithms for IM on Graphs of Bounded Genus

We start with the planar case. Let (G, k) be an instance of Planar-IM where G has n vertices. By Theorem 6.1, we can assume that after an $O(n)$ preprocessing time, the number of vertices n in G satisfies $n \leq 40k$.

Let M be a maximal matching in G and let $I = V(G) - V(M)$. If $V(M)$ contains more than $8k$ vertices, then by contracting each edge of M in $G_M = G(V(M))$ then applying the Four-Color Theorem to G_M, we conclude that G_M, and hence G, has an induced matching of at least k edges, and we can accept the instance (G, k). Assume that $V(M) < 8k$.

The algorithm will look for a set of exactly k edges that form an induced matching. These edges will have at most $2k$ endpoints in $V(M)$. Therefore, we start by enumerating every subset $S \subseteq V(M)$ of size at most $2k$. There are at most $\sum_{i=0}^{2k}\binom{8k}{i}$ such subsets. Let S be one of them. We work under the assumption that every vertex in S is an endpoint of an edge in the induced matching until we either find the desired induced matching, or this assumption turns out to be false. In the latter case we enumerate the next subset S.

If two vertices u and v in S are adjacent, then uv must be an edge in the induced matching; therefore, in this case we include uv, remove every neighbor of u and v from G, and reduce k by 1. After we have included (in the induced matching) every edge whose both endpoints are in S, every remaining vertex in S must be matched with a vertex in I. Observe that if there is a vertex $w \in I$ that is adjacent to at least two vertices in S, then none of the edges joining w to S is in the induced matching. Hence, w could not be an endpoint to an edge in the matching, and w can be removed from I. After removing every such vertex w from I, each remaining vertex in I is adjacent to at most one vertex in S. Now if our original choice of the set S was correct, then by choosing a neighbor in I for every vertex in S, we should obtain an induced matching in G of size k. If such a choice is not possible (for example, a vertex in S does not have a neighbor in I), or the total number of edges in the induced matching at the end of this process is less than k, then our choice of S was incorrect, and we enumerate the next subset S of $V(M)$ of size at most $2k$. After we have enumerated all subsets of $V(M)$ of size at most $2k$, either we have found an induced matching of at least k edges, or no such a matching exists. Noting that there are at most $\sum_{i=0}^{2k} \binom{8k}{i} \leq (2k+1)\binom{8k}{2k}$ such subsets, and that the number of vertices in G is $O(k)$, we have the following theorem:

Theorem 7.1. *The* PLANAR-IM *problem can be solved in* $O(\binom{8k}{2k}k^2 + n) = O(91^k + n)$ *time.*

The above algorithm is a more practical algorithm for small values of the parameter k than the one described in the previous section. In particular, it reduces the problem to a simple enumeration algorithm, as opposed to the previous algorithm which relies on the complicated procedure of separating the planar graph using the Lipton-Tarjan theorem.

We now generalize the result to bounded genus graphs. By Heawood's theorem (Lemma 4.7), the chromatic number of a graph with genus g is bounded by $(7 + \sqrt{1 + 48g})/2$. Thus, a graph on n vertices with genus g has an independent set of at least $2n/(7 + \sqrt{1 + 48g})$ vertices. It follows from the above that if $V(M)$ contains at least $(7 + \sqrt{1 + 48g})k$ vertices, then G has an induced matching of at least k edges. Otherwise, we can enumerate all subsets of $V(M)$ of size at most $2k$ and proceed as before. We conclude with the following theorem.

Theorem 7.2. *The* IM *problem on graphs with n vertices and genus g can be solved in* $O(\binom{(7+\sqrt{1+48g})k}{2k}k^2 + n)$ *time.*

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