Some Unexpected(ly) Open Problems

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Abstract

This note collects open questions and conjectures related to graph drawing research. In spite of their topological provenance, the first three problems are purely combinatorial problems on digraphs and words; only the last problem is a straight-forward crossing number problem.

1 Tournaments and Linear Arrangements

A *tournament* is a directed graph containing exactly one arc between any two vertices. For a tournament T on a set V of vertices let $T(u, v)$ be the number of directed paths of length 2 from u to v. A *tournament arrangement* of a graph $G = (V, E)$ is a tournament T on V, the *cost* of the arrangement is defined as

$$|E| + \sum_{uv \in E} T(u, v).$$

Tournament arrangements generalize *linear arrangements*, in which the tournaments are restricted to be linear orders. We can also view a linear arrangement as a permutation φ mapping V to $\{1, \ldots, |V|\}$; then the value of the linear arrangement is $|E| + \sum_{uv \in E} T(u, v) = \sum_{uv \in E}(T(u, v) + 1) = \sum_{uv \in E} |\varphi(u) - \varphi(v)|$, which is the usual way to define the value of a linear arrangement of the vertices of G, so our definition agrees with the standard definition for tournaments which are linear orders.

Conjecture 1.1 (Pelsmajer, Schaefer, Štefankovič [5])**.** *The minimum cost of a tournament arrangement of a graph is achieved by a linear arrangement.*

This seems a natural conjecture to make, and its truth would have simplified the proof of **NP**-hardness for deciding the independent odd crossing number (a variant of the standard crossing number). For that proof it turned out to be sufficient to show that the conjecture holds for complete graphs:

Theorem 1.2 (Pelsmajer, Schaefer, Štefankovič [5]). *The minimum cost of a tournament arrangement of K_n is $\binom{n+1}{3}$.*

For a minimum linear arrangement of a complete graph, the order of the vertices does not matter, which makes it easy to see that the cost of a minimum linear arrangement is $\binom{n+1}{3}$, verifying the conjecture for complete graphs. The natural next step would be to verify the conjecture for simple families of graphs such as paths, cycles, and trees.

There is a very tempting avenue of attack on the conjecture: Suppose we have a tournament arrangement of a graph G which is not a linear arrangement; then the tournament must contain a directed cycle; find an arc in the tournament so that reversing the arc brings us closer to a linear arrangement (by reducing the number of directed cycles) without increasing the cost of the arrangement. This approach leads into dangerous territory: Ádám conjectured that any directed graph containing a directed cycle contains an edge that can be reversed so as to decrease the number of directed cycles; the conjecture fails for multigraphs, but its status is open for simple graphs. The attack we outlined would require settling Ádám's conjecture for tournaments: any tournament which is not a linear order contains an arc that can be reversed so as to decrease the number of directed cycles; even this variant is still open [1].

2 Edit Distance of Cyclic Words

Given two words, the *swapping distance* between the two words is the smallest number of transpositions of adjacent letters (*swaps*) that turn one word into the other. The swapping distance is a special case of the edit distance problem in which other operations (replace, insert, delete, swap) are allowed for various costs, and it is known to be solvable in time $O(n \log n)$. However, what happens if we ask for the swapping distance of cyclic words, that is words in which the first and last letter are adjacent?

The cyclic variant of the swapping distance models computing the crossing number of a multigraph on two vertices with a given rotation system (a cyclic order of the edges leaving each vertex): the two cyclic words are the orders in which the edges leave the vertices [5]. At this point one would expect to find a (dynamic programming?) algorithm to solve the cyclic swapping distance problem to calculate the crossing number for 2-vertex multigraphs. Instead the reverse is true: we know how to calculate the crossing number in polynomial time using integer programming using relaxation [4], thus giving us an algorithm for computing the swapping distance for cyclic words in which each letter occurs only once, that is, permutations. It seems rather hard to believe that an edit distance problem should require integer programming.

Conjecture 2.1. *The swapping distance of cyclic words can be computed in polynomial time.*

3 Binary Square-Free Words

Thue showed that it is possible to construct a word that does not contain any squares, that is subwords of the form ww. He even showed that an infinite square-free word can be built over the alphabet $\{0, 1, 2\}$. While binary words can be square-free, any binary word of length at least 4 must contain a square, so in particular there are no infinite binary square-free words. However, there are infinite binary words that do not contain any odd squares, that is squares ww where $|w|$ is odd. For example, $010101\ldots$ is such a word. So what about avoiding even squares, squares for which $|w|$ is even? While this problem is harder, one can define such a word $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ explicitly via

$$a_n = \lfloor n\phi \mod 2\rfloor,$$

where $\phi = (\sqrt{5}+1)/2$ is the golden ratio [6]. Then $a_1a_2\ldots = 110001100011\ldots$, a sequence that is listed in Sloane's Encyclopedia of Integer Sequences [2, Sequence A085002].

What is unusual about here is that traditionally square-free words are constructed by repeatedly applying square-free morphisms (morphisms that map square-free words to square-free words) to an initial square-free word.

Question 3.1. Is there a construction of a square-free word over a three-letter alphabet that is not based on recursively applying square-free morphisms?

It is easy to build a square-free word over a four-letter alphabet by combining $\mathbf{a}$ and $010101\ldots$ bit by bit. There are other aspects of the sequence $\mathbf{a}$ which are intriguing; for example, the lengths of the squares that do occur are $1, 5, 21, 89, \ldots$, that is, Fibonacci numbers of the form f_{3n+2}. Numerical evidence seems to suggest that there is a relationship between the length of squares in a sequence $b_n = \lfloor n\alpha \mod 2\rfloor$, for irrational α and the numerators and denominators of the convergents of the continued fraction of α.

4 The Triviality of Adjacent Crossings

In his famous paper on algebraic aspects of the crossing number, Tutte wrote "We are taking the view that crossings of adjacent edges are trivial, and easily got rid of" [7]. Taking Tutte at his word, the following question should be trivial:

> If a graph can be drawn in the plane so that no two non-adjacent edges cross, is the graph planar?

While the answer to the question is yes, the easiest proof seems to involve the Hanani-Tutte theorem which is the stronger statement that a graph is planar if it can be drawn in the plane so that no two non-adjacent edges cross an odd number of times. Hence we ask whether our question above can be answered directly without using the Hanani-Tutte theorem. To emphasize that crossings of adjacent edges are hardly trivial, I propose the following conjecture:

Conjecture 4.1. *If a graph can be drawn in a surface so that no two non-adjacent edges cross, then the graph can be embedded in that surface.*

By the recently established Hanani-Tutte theorem for the projective plane, we know that the conjecture is true for the projective plane, but even the case of the torus remains open [3].

References

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