A Guided Tour of Minimal Indices and Shortest Descriptions

Submitted as a Master's Thesis at the University of Chicago

Marcus Schaefer*

Department of Computer Science
University of Chicago
1100 East 58th Street
Chicago, Illinois 60637, USA
schaefer@cs.uchicago.edu

November 21, 1997

Abstract

The set of minimal indices of a Gödel numbering φ is defined as $\mathrm{MIN}_\varphi = \{e : (\forall i < e)[\varphi_i \neq \varphi_e]\}$. It has been known since 1972 that $\mathrm{MIN}_\varphi \equiv_\mathrm{T} \emptyset''$, but beyond this MIN_φ has remained mostly uninvestigated. This thesis collects the scarce results on MIN_φ from the literature and adds some new observations including that MIN_φ is autoreducible, but neither regressive nor $(1,2)$-computable. We also study several variants of MIN_φ that have been defined in the literature like size-minimal indices, shortest descriptions, and minimal indices of decision tables. Some challenging open problems are left for the adventurous reader.

1 Introduction

How long is the shortest program that solves your problem?

There are at least two ways to interpret this question depending on the type of problem involved. If the program's task is to output one specific object, we are looking for a *shortest description* of that object. This interpretation is closely related to Kolmogorov complexity. Although we have several things to say about shortest descriptions, the main concern of this thesis are programs that compute a function. We will then ask about the complexity of computing a *minimal index* of that function. If we abstract from concrete machine models, the question translates into minimal indices with respect to a numbering of the computable, partial functions[1].

We call e a minimal index (with regard to a given numbering φ) if $\varphi_i \neq \varphi_e$ for all $i < e$. Our main object of study will be the set $\mathrm{MIN}_\varphi = \{e : e \text{ is a minimal index with regard to } \varphi\}$.

The study of minimal indices started in the 1960s with research concentrating on the size of automata. Manuel Blum and Albert Meyer isolated the problem as we know it today, and initiated its

*Partially supported by NSF Grants CCR 92-53582 and CCR-9501794.

[1] Soare [27] suggests a new terminology for computability which will be used throughout this thesis, for example *computable* instead of recursive and *computably enumerable (c.e.)* for recursively enumerable.

research. There is a twofold interest in MIN_φ, one academic and one practical. MIN_φ has served its term as the standard classroom example of a noncomputable set, and surprised many a student (and teacher) by behaving differently from the familiar index sets. Reducing $\emptyset'$ to it is not entirely trivial[2], but until now it was not known why this should be so. We will shed some light on this when we look at bounded reducibilities. Whereas strange behavior is usually welcome in mathematical circles, in the case of MIN_φ it has led to an inexcusable neglect. The only area where minimal indices still receive some attention is computational learning theory, and here MIN_φ is of practical interest. This line of research was started by Freivalds and Kinber in the 1970s and continues to the present day [4]. The underlying motivation is to not to be content with just learning a function, but to find a program that is not too much longer than the shortest possible.

A generalization of the set MIN_φ as defined above was suggested in the early seventies, at least partially inspired by Blum's investigations into the axiomatization of complexity measures [1]. The idea was to allow size-measure on the indices; instead of saying i is smaller than e if $i < e$, we use a size measure s to decide; we say i has smaller size than e if $s(i) < s(e)$. These size measures s measure anything from time to space complexity, not just program length [29]. Some problems connected with the generalized version of MIN_φ were solved by Bagchi [1] and Pager [25], but many remained, and still remain, open.

The thesis starts with the investigation of MIN_φ and then traces the variants considered in the literature. To the extent of my knowledge this is a fairly complete account with the exception of the learning-theoretic cousins of MIN_φ.

For reducibilities and other notation see the standard references [21, 28].

2 Minimal Indices and Shortest Descriptions

2.1 Definitions

Given a numbering $(\varphi_i)_{i\in\omega}$ we call an index e of the computable function φ_e *minimal* if φ_e is different from all functions with smaller index in the numbering.

Definition 2.1 *Let φ be a Gödel numbering. Define*

$$\mathrm{MIN}_\varphi = \{e : (\forall i < e)[\varphi_i \neq \varphi_e]\},$$

the set of minimal indices *of φ.*

A Gödel numbering is an effective numbering φ of all computable partial functions such that for every effective numbering ψ a φ-index can be computed from a ψ-index. The definition of MIN_φ restricts φ to Gödel numberings, since they are the natural programming systems. This is also witnessed by the behavior of MIN_φ itself. If we allowed arbitrary numberings of the computable, partial functions, MIN_φ could be almost any set. And even for effective numberings there is the pathological Friedberg numbering [7] for which MIN_φ would be equal to ω, i.e., every index would be minimal.

Belonging to MIN_φ is the function $\min_\varphi$ which computes the minimal φ-index of a program. More formally the function is defined as $\min_\varphi(i) = (\mu e)[\varphi_i = \varphi_e]$. Then $\mathrm{MIN}_\varphi \leq_{\mathrm{T}} \min_\varphi$, since $e \in \mathrm{MIN}_\varphi$ if and only if $\min_\varphi(e) = e$. Thus determining the complexity of MIN_φ will give us a lower bound (and as it turns out the exact degree) of the task of computing the minimal index of a function.

[2] And the reader is invited to try his or her hand before proceeding to the next section.

If instead of computable functions we are interested in finite objects (which are represented as numbers), we can ask for the shortest description of that object in a given Gödel numbering. This borders on the realm of Kolmogorov complexity.

Definition 2.2 *Let φ be a Gödel numbering. Define*

$$\mathrm{R}_\varphi = \{e : (\forall i < e)[\varphi_i(0) \neq \varphi_e(0)]\},$$

the set of shortest descriptions *of φ.*

From now on we will drop the φ in MIN_φ and R_φ and simply write MIN and R if we think of φ as a fixed, but arbitrary Gödel numbering. We will write out MIN_φ and R_φ to stress the dependency on φ. The same policy applies to other objects from computability theory like $\emptyset'$, TOT, or W_e which depend on the particular Gödel numbering relative to which they are defined, we will assume that Gödel numbering to be the same as the one used to define MIN. This is not only a natural assumption, it can also be made without any loss of generality, since the results will easily translate to other versions of $\emptyset'$, TOT, or W_e.

2.2 Immunity

Perhaps the first result on minimal indices can be found in Manuel Blum's 1967 paper on machine size [2]. His Theorem 1 in a slightly modernized and restricted version states that the set MIN is *immune*, that is, it does not contain an infinite c.e. subset (and is infinite itself).

Theorem 2.3 (Blum [2]) MIN *is immune.*

Proof. Suppose there is a computable function f such that the range of f is an infinite subset of MIN_φ. Define the computable function $g(e) = f((\mu i)[f(i) > e])$. Then $g(e) > e$ for all e, and, since $g(e) \in \mathrm{MIN}_\varphi$, we have $\varphi_{g(e)} \neq \varphi_e$ for all e, contradicting the Recursion Theorem. □

This proof is probably the easiest way of showing that MIN is not computable. The usual attempts of m-reducing $\emptyset'$ or $\overline{\emptyset'}$ to MIN are doomed, as shown in Theorem 2.10.

By a result of Case [22] the set of minimal indices is even strongly effectively immune. Owings gives a proof of a more general result in the same paper. We include the short, but instructive proof of the original result.

Definition 2.4 *A set A is* strongly effectively immune *if it is infinite and there is a computable function f such that*

$$(\forall e)[W_e \subseteq A \Rightarrow \max(W_e) \leq f(e)].$$

Theorem 2.5 (Case [22, Theorem 1]) MIN *is strongly effectively immune.*

Proof. Fix φ, and let $W_e = \mathrm{dom}(\varphi_e)$ for all e. Using the Recursion Theorem define a computable function f fulfilling:

$$\varphi_{f(e)}(x) = \begin{cases} \varphi_i(x) & \text{if } \langle i, s\rangle \text{ is the smallest } \langle i', s'\rangle \text{ with } i' \in W_{e,s'} \text{ and } i' > f(e), \\ \uparrow & \text{otherwise.} \end{cases}$$

Now assume $W_e \subseteq \mathrm{MIN}_\varphi$. If there was some $i \in W_e$ with $i > f(e)$, then $\varphi_{f(e)} = \varphi_i$ for some such i. Since $i \in W_e$ it is the minimal index of the function φ_i which contradicts $i > f(e)$. Hence W_e contains only elements which are at most $f(e)$. Note that the proof did not depend on W_e being defined using φ. Any other Gödel numbering would have done just as well. □

Although we cannot enumerate an infinite number of minimal indices we can compute an infinite number of disjoint sets which contain a minimal index, in other words MIN fails to be hyperimmune. Recall that $(D_e)_{e\in\omega}$ is the canonical numbering of the finite sets.

Definition 2.6 *A set A is called* hyperimmune *if it is infinite and there is no computable function f such that*

- $(D_{f(i)})_{i\in\omega}$ *is a family of pairwise disjoint sets, and*
- $D_{f(i)} \cap A \neq \emptyset$.

Lemma 2.7 (Meyer [20]) MIN *is not hyperimmune.*

Proof. To see this, we will define a strictly increasing function $\alpha(n)$ inductively such that the pairwise disjoint sets $I_n = \{x : \alpha(n) < x \leq \alpha(n+1)\}$ intersect MIN_φ for every n. Start with $\alpha(0) = 0$. Let $i_0, \ldots, i_{\alpha(n)+1}$ be φ-indices of the constant functions $(\lambda x).[0], \ldots, (\lambda x).[\alpha(n)+1]$. Such indices can be found effectively. Let $\alpha(n+1)$ be the maximum of $i_0, \ldots, i_{\alpha(n)}$. Then the interval $\{x : x \leq \alpha(n+1)\}$ contains φ-indices of at least $\alpha(n)+2$ functions. Since $\{x : x \leq \alpha(n)\}$ contains at most $\alpha(n)+1$ elements, there is a minimal index in $\{x : \alpha(n) < x \leq \alpha(n+1)\}$. □

The gap between immune and hyperimmune is large, and by introducing a new notion of immunity, we can give a stronger characterization of MIN which will be useful in the study of bounded reducibilities.

Definition 2.8 (Fenner and Schaefer [6]) *A set A is called k-*immune *if it is infinite and there is no computable function f such that*

- $(D_{f(i)})_{i\in\omega}$ *is a family of pairwise disjoint sets, and*
- $D_{f(i)} \cap A \neq \emptyset$, *and*
- $|D_{f(i)}| \leq k$.

A set A is called ω-immune *if it is k-immune for every k.*

Note that 1-immunity is the same as immunity and hyperimmunity implies ω-immunity. One reason for the lack of interest in ω-immune sets is the folklore result that a set which is immune and co-c.e. is ω-immune [6]. Therefore we do not get any new notions of simplicity between simple and hypersimple.

Our interest in k-immunity here is explained by the following result.

Theorem 2.9 MIN *is ω-immune.*

Proof. The proof is a generalization of the immunity proof using the k-fold Recursion Theorem. Suppose MIN was not k-immune. Let $D_{f(i)}$ witness this as in the definition of k-immunity. Define a computable function $h(x_1, \ldots, x_k) = f((\mu i)[(\forall z \in D_{f(i)})(\forall j)[z > x_j]])$. The function h picks out the index of the first set in the enumeration for which all elements are bigger than any x_j. We use h to define k computable functions. For $1 \leq i \leq k$ let

$$g_i(x_1, \ldots, x_k) = \text{the } i\text{th element of } D_{h(x_1,\ldots,x_k)} \,.$$

By the k-fold Recursion Theorem there are k indices $e_1, \ldots, e_k$ such that $\varphi_{g_i(e_1,\ldots,e_k)} = \varphi_{e_i}$ for all $1 \leq i \leq k$. Since $g_i(e_1, \ldots, e_k) > e_i$ this contradicts the fact that $g_i(e_1, \ldots, e_k)$ has to be a minimal index for some i. □

A careful look at the proof shows that MIN is in fact effectively k-immune in the following sense: there is a total computable function g such that if W_e is a set of canonical indices of pairwise disjoint sets, all of which intersect MIN and contain at most k elements, then $g(e)$ is an upper bound on the cardinality of W_e.

By the last theorem MIN is a natural example of a set which is not hyperimmune, although it is ω-immune. More on the realm between ω-immunity and hyperimmunity can be found in the paper by Fenner and Schaefer [6].

2.3 The Degrees of MIN

There is a strong connection between immunity properties and completeness under strong reducibilities [6] which allows us to exploit the results of the last section.

The immunity of MIN immediately implies that no c.e., noncomputable set conjunctively reduces to MIN, and $\overline{K}$ does not m-reduce to MIN. It can also be shown that no c.e., noncomputable set bounded disjunctively reduces to MIN.

A more interesting result is the following which was first proved by Fenner and Schaefer.

Theorem 2.10 (Fenner and Schaefer [6]) $\emptyset' \not\leq_{\mathrm{btt}}$ MIN.

The proof will adapt a result of Denisov's according to which no hyperimmune set can be truth-table complete.

Proof. Define $A = \{e : \varphi_e(0) = 0\}$ and $B = \{e : \varphi_e(0) = 1\}$. It is a standard exercise to show that these two disjoint c.e. sets are not separated by any computable set. Assume $\emptyset' \leq_{\mathrm{btt}}$ MIN. Then $A \leq_{\mathrm{btt}}$ MIN. Let the computable function $f : \omega \to \omega^k$ and the computable k-ary truth-table α_x witness the reduction, i.e.,

$$x \in \mathrm{MIN} \iff \alpha_x(\chi_{\mathrm{MIN}}(f(x))) = 1,$$

where $\chi_{\mathrm{MIN}}(x_1, \ldots, x_k)$ is the *characteristic vector* $(\chi_{\mathrm{MIN}}(x_1), \ldots, \chi_{\mathrm{MIN}}(x_k))$ of MIN. We will prove that for every n we can effectively find a set of at most $2k$ elements all of which are greater than n and one of which lies in MIN. This immediately implies that MIN would not be $2k$-immune which contradicts MIN's ω-immunity. To prove the claim consider two cases.

First suppose that for all n there are $x \in A$ and $y \in B$ such that $\alpha_x(\chi_D(f(x))) = \alpha_y(\chi_D(f(y)))$ for all $D \subseteq \{0, \ldots, n\}$. In particular the equality will hold true for $D = \mathrm{MIN} \cap \{0, \ldots, n\}$. Since $x \in A$ and $y \in B$ we know that $\alpha_x(\chi_{\mathrm{MIN}}(f(x))) \neq \alpha_y(\chi_{\mathrm{MIN}}(f(y)))$. This means that $\mathrm{MIN} \cap (f(x) \cup f(y))$ and $D \cap (f(x) \cup f(y))$ have to disagree, hence it is enough to let the finite set we are looking for be $F = (f(x) \cup f(y)) - \{0, \ldots, n\}$.

In the second case we know that there is an n such that for all $x \in A$ and $y \in B$ there is a $D \subseteq \{0, \ldots, n\}$ for which $\alpha_x(\chi_D(f(x))) \neq \alpha_y(\chi_D(f(y)))$. Therefore we can partition the integers according to the values of $(\alpha_x(\chi_D(f(x)))_{D \subseteq \{0,\ldots,n\}}$. This yields finitely many equivalence classes which respect A and B by assumption. Therefore if we take C to be the union of all the parts of the partition which intersect B, then C is a computable set that separates A and B contradicting their inseparability. □

Remark. A closer inspection of the proof reveals that it establishes that $\emptyset'$ does not k-tt-reduce to a $2k$-immune set. Fenner and Schaefer [6] showed that in fact $\emptyset'$ does not k-tt-reduce to a k-immune set, a result which is tight, but more difficult to prove, and the present version is sufficient to deal with *btt*-reductions.

The above shows that any reduction of $\emptyset'$ to MIN will not be trivial. As a matter of fact the best result known is a wtt-reduction. Recall that a Turing reduction is called a *wtt-reduction* if there is a computable bound on the queries asked to the oracle.

Theorem 2.11 (Meyer [20]) $\emptyset' \leq_{\mathrm{wtt}}$ MIN.

Proof. Fix a Gödel numbering φ. We will show that $\emptyset'_\varphi \leq_{\mathrm{wtt}} \mathrm{MIN}_\varphi$, where $\emptyset'_\varphi = \{i : \varphi_i(i)\downarrow\}$. Since $\emptyset' \leq_{\mathrm{m}} \emptyset'_\varphi$ this will conclude the proof.

Let a be the minimal index of the function that is undefined everywhere. Define a computable function f as follows.

$$\varphi_{f(i)}(x) = \begin{cases} 0 & \text{if} \quad \varphi_{i,x}(i)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

To decide whether $i \in \emptyset'_\varphi$ do the following: for every $e \in \mathrm{MIN}_\varphi \cap \{0, \ldots, f(i)\} - \{a\}$ dovetail φ_e on all integers to find some x_e for which $\varphi_e(x_e)$ is defined. Note that all searches terminate, since we excluded a. Now let x be the maximum of the x_e. Then $i \in \emptyset'_\varphi$ if and only if $\varphi_{i,x}(i)\downarrow$: the minimal index of the function $\varphi_{f(i)}$ belongs to the set $\mathrm{MIN}_\varphi \cap \{0, \ldots, f(i)\}$, so if $i \in \emptyset'_\varphi$, then x is an upper bound on the first argument, on which the function $\varphi_{f(i)}$ is defined. □

Corollary 2.12 *With* MIN_φ *as an oracle, we can compute the minimal index of a function given a* φ*-index of that function. That is, for* $\min_\varphi$ *defined as* $\min_\varphi(i) = (\mu e)[\varphi_i = \varphi_e]$ *we have* $\min_\varphi \leq_{\mathrm{T}} \mathrm{MIN}_\varphi$.

Proof. By the proof of the preceding theorem we know that $\emptyset'_\varphi \leq_{\mathrm{T}} \mathrm{MIN}_\varphi$. Hence with a MIN_φ oracle we can effectively in i and x decide, whether $\varphi_i(x)$ diverges. Given an index i, we compute initial segments of all φ_j with $j \in \mathrm{MIN}_\varphi \cap \{0, \ldots, i\}$ (including the undefined values), until all but one of them is different from φ_i. The index of this function is the minimal index of φ_i. □

We observed earlier that the reverse is also true, namely $\mathrm{MIN}_\varphi \leq_{\mathrm{T}} \min_\varphi$, hence MIN_φ and $\min_\varphi$ have the same Turing degree.

Theorem 2.13 (Meyer [20, Theorem 7]) $\emptyset'' \leq_{\mathrm{T}} \mathrm{MIN}$.

Proof. Fix a Gödel numbering φ. Since $\emptyset'' \leq_{\mathrm{m}} \mathrm{TOT}_\varphi$, it will be sufficient to show that $\mathrm{TOT}_\varphi \leq_{\mathrm{T}} \mathrm{MIN}_\varphi$.

Let a be the minimal index of the function that is zero everywhere. Define a computable function f as follows.

$$\varphi_{f(i)}(x) = \begin{cases} 0 & \text{if } \varphi_i(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Now $i \in \mathrm{TOT}_\varphi$ iff $\varphi_{f(i)}(x) = 0$ for all x iff $\min_\varphi(f(i)) = a$, which is decidable in MIN_φ by the preceding corollary. □

The last theorem has several immediate consequences:

Lemma 2.14

(*i*) MIN $\equiv_{\mathrm{T}} \emptyset''$, *because* MIN *is in* Σ^0_2.

(*ii*) *All* MIN_φ *are Turing equivalent.*

(*iii*) MIN *is not in* Π^0_2 *(since it is in* Σ^0_2*).*

(*iv*) MIN *is not introreducible, i.e., there is an infinite subset of* MIN *to which* MIN *does not Turing reduce. This follows because* MIN *as a* Σ^0_2 *set has an infinite subset computable in* $\emptyset'$.

The second observation leads to the question (first asked by Meyer) whether all MIN_φ are tt-equivalent. Two partial results to this question have been obtained so far.

Theorem 2.15 (Kinber [11]) *There are two Gödel numberings* φ, ψ *such that* MIN_φ *and* MIN_ψ *are incomparable with regard to btt-reductions.*

Marandžjan proves the same result for conjunctive reductions (c-reductions).

Theorem 2.16 (Marandžjan [18, 19]) *There are two Gödel numberings* φ, ψ *such that* MIN_φ *and* MIN_ψ *are incomparable with regard to c-reductions.*

This leaves us with the possibility that the tt-degree of MIN_φ will depend on φ. Kinber [11] mentions that it is possible to construct a Gödel numbering, for which MIN_φ is tt-complete for Σ^0_2, and Marandžjan provides a proof which shows that MIN_φ can be made d-complete for Σ^0_2. With some more care we can even construct a Kolmogorov numbering such that MIN_φ is d-complete for Σ^0_2. Remember that a Gödel numbering φ is a Kolmogorov numbering if for every Gödel numbering ψ there is a linearly bounded computable function that transforms ψ-indices into φ-indices. It is well known that Kolmogorov numberings exist [26, Theorem 1].

The theorem gives us a tight result with regard to disjunctive reductions, since (as we mentioned earlier) not even $\emptyset'$ *bd*-reduces to MIN_φ.

Theorem 2.17 *There is a Kolmogorov numbering φ such that MIN_φ is d-complete for Σ_2^0.*

Proof. Fix a Kolmogorov numbering ψ. We will construct a numbering φ by alternately coding TOT_ψ, so it can be recovered by a tt-reduction, and copying parts of ψ, so φ will become a Kolmogorov numbering itself.

The construction of φ will proceed in stages. At stage s all functions of index less than $w(s)$ have been defined. Exactly $i(s)$ of these have been copied from the Kolmogorov numbering ψ. The other functions are for coding purposes. The two primitive recursive functions w and i are defined as follows: $w(0) = i(0) = 0$. The induction is:

$$\begin{aligned} i(s+1) &= i(s) + w(s) + 2(i(s)+1) \\ w(s+1) &= 2[w(s) + 2(i(s)+1)] \end{aligned}$$

This means that in stage s of the construction $w(s)+2(i(s)+1)$ functions are copied from ψ and $2(i(s)+1)$ functions are used for the coding. Note that it is obvious from the definition that $w(s) \leq 2i(s)$ for all s.

Construction of φ.

Stage s. (Define φ_i for $w(s) \leq i < w(s+1)$.)

***Step 1.* (Code $s \in \mathrm{TOT}_\psi$.)** For $2i$ with $w(s) \leq 2i < w(s) + 2(i(s)+1)$ let

$$\begin{aligned} \varphi_{2i}(x) &= \begin{cases} i & \text{if} \quad \psi_s(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases} \\ \varphi_{2i+1}(x) &= i \qquad \text{for all} \quad x. \end{aligned}$$

***Step 2.* (Copy $\psi_{i(s)}$ up to $\psi_{i(s+1)-1}$ into φ.)** For i with $w(s) + 2(i(s)+1) \leq i < w(s+1)$ define

$$\varphi_i = \psi_{i(s)+i-[w(s)+2(i(s)+1)]}.$$

End of Construction.

Two lemmata conclude the proof of the theorem.

Lemma 1 $\overline{\mathrm{TOT}} \leq_{\mathrm{d}} \mathrm{MIN}_\varphi$.

Proof. We claim that

$$s \in \mathrm{TOT}_\psi \iff \{w(s)+1, w(s)+3, \ldots, w(s)+2i(s)+1\} \cap \mathrm{MIN}_\varphi = \emptyset.$$

One direction is immediate: if $s \in \mathrm{TOT}_\psi$, then $\psi_s(x)$ is defined for all x. Hence $\varphi_{2i} = \varphi_{2i+1}$ for all $2i$ with $w(s) \leq 2i < w(s) + 2(i(s)+1)$, so $\{w(s)+1, w(s)+3, \ldots, w(s)+2i(s)+1\} \cap \mathrm{MIN}_\varphi = \emptyset$.

Assume $s \notin \mathrm{TOT}_\psi$. This means that all functions φ_i with $i \in \{w(s)+1, w(s)+3, \ldots, w(s)+2i(s)+1\}$ are different from all functions φ_j where $w(s) \leq j < i$. Therefore $i \in \{w(s)+1, w(s)+3, \ldots, w(s)+2i(s)+1\}$ can only be nonminimal if φ_i agrees with some φ_j where $j < w(s)$. Furthermore the functions at odd indices that are added in Step 1 of the construction are pairwise different. For such an index to be nonminimal its function has to agree with a function added in Step 2 of the construction. Up to stage s only $i(s)$ many functions have been added during Step 2. That means one of the $i(s)+1$ functions φ_i with $i \in \{w(s)+1, w(s)+3, \ldots, w(s)+2i(s)+1\}$ must be minimal.

This proves that $\overline{\mathrm{TOT}_\psi} \leq_{\mathrm{d}} \mathrm{MIN}_\varphi$. Since $\overline{\mathrm{TOT}} \leq_{\mathrm{m}} \overline{\mathrm{TOT}_\psi}$ this concludes the proof of the lemma. □

Lemma 2 *φ is a Kolmogorov numbering.*

Proof. The construction starts with a Kolmogorov numbering ψ. We will show that the construction above stretches out ψ only by a factor, and therefore is still a Kolmogorov numbering.

We can rewrite Step 2 of the construction as follows:

For k with $i(s) \leq k < i(s+1)$ define $\varphi_{w(s)+2(i(s)+1))+k} = \psi_{i(s)+k}$.

This shows two things: φ is a Gödel numbering, since it includes all functions enumerated by ψ, and secondly the φ-index of a function is within a linear function of its ψ-index:

$$w(s) + 2(i(s) + 1) + k \leq 4(i(s) + k) + 2$$

using $w(s) \leq 2i(s)$, which is immediate from the definitions of w and i. □

MIN_φ is d-complete for Σ_2^0 by the first lemma, where φ is a Kolmogorov numbering by the second lemma. This concludes the proof of the theorem. □

2.4 Weak Notions of Computability and Enumerability

We have already seen that MIN is difficult to compute, since it is complete for the second level of the arithmetical hierarchy. Is it possible for MIN to be computable or enumerable in some weaker sense? In this section we suggest that the answer is no, although the reader should compare this to the result on autoreducibility in the next section.

Perhaps the most famous notion of approximate computability is semirecursiveness as introduced by Jockusch in 1968 [8]. A set A is called *semirecursive* if there is a total computable function f in two arguments such that $f(a, b) \in A \cap \{a, b\}$ if $A \cap \{a, b\}$ is not empty.

MIN is not semirecursive. This follows from an easy general result by Jockusch [8]: every immune and semirecursive set is hyperimmune. Since MIN is immune without being hyperimmune, it cannot be semirecursive.

Semirecursiveness is generalized by the notion of $(1, k)$-computability which originated in frequency computation, an area closely related to the theory of bounded queries. Frequency computation is another attempt at introducing a notion of approximate computability; for a recent paper on the subject see Kummer and Stephan [16]. Let χ_A denote the characteristic function of A. Then the *characteristic vector* $\chi_A(x_1, \ldots, x_k)$ is defined as $(\chi_A(x_1), \ldots, \chi_A(x_k))$.

Definition 2.18 *A set A is said to be* $(1, k)$-computable *if there is a computable total function $f : \omega^k \to \{0,1\}^k$ such that for all $x_1 < \cdots < x_k$ the characteristic vector $\chi_A(x_1, \ldots, x_k)$ and $f(x_1, \ldots, x_k)$ are different.*

If A is $(1, k)$-computable for some k, it is called approximable.

Suppose we have A and f as in the definition. Why would A be called $(1, k)$-computable? Instead of the k-bit vector $f(x_1, \ldots, x_k)$ consider the vector obtained by flipping all k bits. Denote this vector by $\overline{f}(x_1, \ldots, x_k)$. Then $\overline{f}(x_1, \ldots, x_k)$ agrees with the characteristic vector $\chi_A(x_1, \ldots, x_k)$ in at least one bit: we can effectively answer one out of k queries to A correctly. This is the original definition of $(1, k)$-computability, but we find the definition given above more convenient.

It would be surprising if MIN was approximable, but unfortunately we have not been able to show that this is not the case. There are two partial results, however. First we show that MIN_φ is not approximable for some Gödel numbering, and complement this by a result which implies that MIN_φ is not $(1, 2)$-computable for any Gödel numbering.

Theorem 2.19 *There is a Gödel numbering φ such that MIN_φ is not approximable.*

Proof. The proof will be a straightforward diagonalization construction of φ. We will only prove the existence of a φ, for which MIN_φ is not $(1,k)$-computable for a fixed k, and argue that the construction can be easily adjusted to ensure nonapproximability.

Fix a Gödel numbering ψ. Call a function f a *potential* $(1,k)$*-operator* if f is total and takes on values in $\{0,1\}^k$.

The construction will meet the requirements:

R_n : if ψ_n is a potential $(1,k)$-operator, then there are $x_1 < \ldots < x_k$ such that $\psi_n(\langle x_1,\ldots,x_k\rangle) = \chi_{\mathrm{MIN}_\varphi}(x_1,\ldots,x_k)$.

Let $\alpha(0)=0$, and $\alpha(n+1) = (\alpha(n)+1) + k2^k(\alpha(n)+2)$.

For all n we will let $\varphi_{\alpha(n)} = \psi_n$. This guarantees that φ is a Gödel numbering. The indices between $\alpha(n)$ and $\alpha(n+1)$ will be used to satisfy R_n. To this end we split up the interval $I_n = \{z : \alpha(n) < z < \alpha(n+1)\}$ into $2^k(\alpha(n)+2)$ blocks of size k. That is for $i = 1,\ldots,2^k(\alpha(n)+2)$ define

$$I_n^i = \{z : \alpha(n) + (i-1)k < z \le \alpha(n) + ik\},$$

so the blocks I_n^i partition I_n.

Computation of φ_e.

Case 1. **(Make φ a Gödel numbering.)** If $e = \alpha(n)$ for some n, then define $\varphi_e = \psi_n$.

Case 2. **(Diagonalize.)** Determine the unique n, i, j and z such that e is the j-th element in the interval $I_n^i = \{z,\ldots,z+k-1\}$. Compute $\psi_n(\langle z, z+1,\ldots,z+k-1\rangle)$. If the computation terminates with $v = (v_1\ldots v_k) \in \{0,1\}^k$, then do the following: if $v_j = 0$, then $\varphi_e = \varphi_0$, else let φ_e be the function that outputs e on every input.

To show that the numbering φ so constructed yields a MIN_φ which is not $(1,k)$-computable, it is sufficient to prove that all R_n are fulfilled.

Assume φ_n is a potential $(1,k)$-operator. Then the computation of φ_n on the $2^k(\alpha(n)+2)$ blocks of I_n must converge. Since there are only 2^k different k-bit vectors, φ_n has to take on some value $v \in \{0,1\}^k$ on at least $\alpha(n)+2$ blocks. We claim that for one of these blocks v and the characteristic vector on this block agree. The zeroes in v are not a problem, since φ_0 is copied, making the corresponding index nonminimal. For a 1 in v we compute a constant function, which is different from any other function computed for the same purpose in any other block. Hence there are only $\alpha(n)+1$ functions (namely those with indices in $\{z : z \le \alpha(n)\}$) which could possibly agree with the constant functions. Since there are $\alpha(n)+2$ blocks, there is one block, for which every constant function computed in that block is minimal. Then v is the characteristic vector on this block, diagonalizing the potential $(1,k)$-operator φ_n.

Finally we note that there was nothing requiring us to make k constant, so we can diagonalize against all potential $(1,k)$-operators, for all k at the same time, yielding the general result. □

Although we were unable to show that the preceding result holds true for all Gödel numberings, we have been able to obtain a result generalizing $(1,2)$-computability in another direction.

Definition 2.20 (Kummer and Stephan [15]) *A set A is called* $(3,2)$-verbose *if there is a computable function f such that $\chi_A(x_1,x_2) \in W_{f(x_1,x_2)}$ and $|W_{f(x_1,x_2)}| \le 3$ for all x_1, x_2.*

$(3,2)$-verboseness comprises several other familiar (and less familiar) properties several of which formalize weak notions of enumerability.

Fact 2.21 *If a set A has any of the following properties then it is $(3,2)$-verbose:*

- $(1,2)$-*computable,*
- *semirecursive* [8],
- *semi-c.e.* [9], *i.e., there is a computable partial function f such that $f(a,b) \in A \cap \{a,b\}$ whenever $A \cap \{a,b\}$ is not empty,*
- *weakly semirecursive* [9], *i.e., there is a computable partial function f such that $f(a,b) \in A \cap \{a,b\}$ whenever $|A \cap \{a,b\}| = 1$,*
- *regressive, i.e., there is a computable partial function f and an enumeration $a_0, a_1, \ldots$ of A without repetition (but not necessarily effective) such that $f(a_0) = a_0$ and $f(a_{n+1}) = a_n$.*

Hence the next theorem tells us that MIN is neither regressive, nor $(1,2)$-computable, nor (weakly) semirecursive, nor semi-c.e. The fact that MIN is not regressive was first shown by Fenner using a different proof.

Theorem 2.22 MIN *is not $(3,2)$-verbose.*

For the proof we will use the following lemma about MIN.

Lemma 2.23 *There are sets $A, B \leq_{\mathrm{T}} \emptyset'$ such that $A \subseteq \mathrm{MIN} \subseteq \overline{B}$, and A and B are not separated by a co-c.e. set, i.e., no C with $A \subseteq C \subseteq \overline{B}$ is co-c.e.*

Proof. Fix the Gödel numbering φ. Let $F = \{e : (\forall n > 0)[\varphi_e(n) \uparrow]\}$. Then $F \leq_{\mathrm{T}} \emptyset'$. Let $A = \mathrm{MIN} \cap F$ and $B = \overline{\mathrm{MIN}} \cap F$. Then $A, B \leq_{\mathrm{T}} \emptyset'$, since using a $\emptyset'$ oracle we can find out whether an index in F is a minimal index. Suppose $A \subseteq C \subseteq \overline{B}$, so $\mathrm{MIN} \cap F \subseteq C \subseteq \mathrm{MIN} \cup \overline{F}$. We claim that in this case C is not co-c.e., which finishes the proof.

To show that the claim is true, define a computable function f by

$$\varphi_{f(e)}(x) = \begin{cases} (\mu s)[e \in \emptyset'_s] & \text{if } e \text{ is in } \emptyset' \text{ and } x = 0 \\ \uparrow & \text{otherwise,} \end{cases}$$

where $(\emptyset'_s)_{s\in\omega}$ is a computable enumeration of $\emptyset'$. Note that $f(e) \in F$ for all e. If $e \in \emptyset'$, the first value of $\varphi_{f(e)}$ contains the first stage at which e is enumerated into $\emptyset'$, otherwise $\varphi_{f(e)}$ is the function that is undefined everywhere. Let a be the minimal index of $(\lambda x)[\uparrow]$. For every $i \in C \cap \{0, \ldots, f(e)\} - \{a\}$ there is an x_i such that $\varphi_i(x_i)$ is defined. This is true, since i is either a minimal index of φ_i (in which case since $i \neq a$ the function φ_i has to be defined somewhere), or i is not in F, which means that φ_i is defined for some $n > 0$.

Assume that C is co-c.e. For each i in $D = \{0, \ldots, f(e)\} - \{a\}$ start searching for an x_i as above. Simultaneously enumerate $\overline{C}$ and eliminate elements appearing in $\overline{C}$ from D. Then at some finite stage, D will only contain indices for which witnesses x_i have been found. With this D compute $m(e) = \max\{\varphi_i(x_i) : i \in D\}$. Then $e \in \emptyset'$ if and only if $e \in \emptyset'_{m(e)}$ contradicting that $\emptyset'$ is not computable. □

Proof of Theorem 2.22. Suppose MIN is $(3,2)$-verbose via the computable function f, i.e., $\chi_{\mathrm{MIN}}(x,y) \in W_{f(x,y)}$ and $|W_{f(x,y)}| \leq 3$ for all x and y. Let A and B be chosen as in the lemma. There are two cases.

- There is $x \notin \mathrm{MIN}$ such that $(\forall y \in A)[(1,1) \in W_{f(x,y)}]$ and $(\forall y \in B)[(1,0) \in W_{f(x,y)}]$.

Fix such an x, and define $C = \{y : (1,0) \notin W_{f(x,y)} \text{ or } (0,0) \notin W_{f(x,y)}\}$. Obviously C is a co-c.e. set. We prove that C separates A and B which contradicts the choice of A and B. If y is in A, then by assumption $(1,1) \in W_{f(x,y)}$. Furthermore $(0,1) \in W_{f(x,y)}$ since it is the correct characteristic vector. Since $W_{f(x,y)}$ contains at most three elements one of $(1,0)$ or $(0,0)$ can not be in $W_{f(x,y)}$, whence $y \in C$. This proves that $A \subseteq C$. Now assume that $y \in B$. Then $(0,0) \in W_{f(x,y)}$ (since it is the correct characteristic vector) and $(1,0) \in W_{f(x,y)}$ (by assumption). Then $y \notin C$ by definition, proving that $B \subseteq \overline{C}$.

- For all $x \notin \text{MIN}$ either $(\exists y \in A)[(1,1) \notin W_{f(x,y)}]$ or $(\exists y \in B)[(1,0) \notin W_{f(x,y)}]$.

  In this case we have a Σ_2 witness for $x \notin \text{MIN}$.

$$x \in \text{MIN iff } (\forall y \in A)[(1,1) \in W_{f(x,y)}] \text{ and} (\forall y \in B)[(1,0) \in W_{f(x,y)}].$$

  The implication from left to right holds because $W_{f(x,y)}$ has to contain the correct characteristic vector. The other direction uses the fact that if $x \notin \text{MIN}$ there is either a $y \in B$ for which $(1,0) \notin W_{f(x,y)}$ or a $y \in A$ such that $(1,1) \notin W_{f(x,y)}$. Thus $x \in \text{MIN}$ is equivalent to a formula that is Π^0_1 in $A \oplus B \equiv_{\text{T}} \emptyset'$, hence MIN is in Π^0_2, which we know to be false by Lemma 2.14.

□

I conjecture that MIN is not approximable. In fact, all that would be necessary to prove this conjecture is to show that there are $A, B \in \Sigma^0_2$ such that $A \subseteq \text{MIN} \subseteq \overline{B}$ and A and B are not separated by a set computable in $\emptyset'$, i.e., for all C with $A \subseteq C \subseteq \overline{B}$, $C \not\leq_{\text{T}} \emptyset'$. By relativizing a theorem of Kummer and Stephan [16, Theorem 3.2] to $\emptyset'$ we would then have that MIN is not even $(1,k)$-computable by a function computable in $\emptyset'$.

2.5 Autoreducibility

Most of the results concerning MIN are of a negative character, due to its extreme thinness. However, there is at least one nontrivial property MIN does have: it is autoreducible, namely there is an oracle Turing machine which can decide whether $e \in \text{MIN}$ by making queries to MIN which are different from e.

The proof will be a modification of the proof that MIN is Turing complete for $\emptyset''$. We first need a lemma:

Lemma 2.24 *Let φ be a Gödel numbering. Given i, x and a finite set $D \subset \omega$, we can effectively decide whether $\varphi_i(x)$ diverges by using* MIN_φ *as an oracle without asking any element of D.*

Proof. Fix a Gödel numbering φ. Let a be the minimal index of the function that is undefined everywhere. Define computable functions f_j by

$$\varphi_{f_j(i)}(s) = \begin{cases} j & \text{if} \quad \varphi_{i,s}(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases},$$

where $j \in \{0,\ldots,|D|\}$. Let $f(i) = \max\{f_j(i) : j \in \{0,\ldots,|D|\}\}$. To decide whether $\varphi_i(x)\downarrow$, do the following: for every $e \in \text{MIN}_\varphi \cap \{0,\ldots,f(i)\} - (D \cup \{a\})$ dovetail φ_e on all integers to find some s_e for which $\varphi_e(s_e)$ is defined. Note that all searches terminate, since we excluded a. Now let s be the maximum of the s_e. Then $\varphi_i(x)\downarrow$ if and only if $\varphi_{i,s}(x)\downarrow$. The reason is that if $\varphi_i(x)$ converges, then the minimal index of at least one of the $|D|+1$ functions f_j is in $\text{MIN}_\varphi \cap \{0,\ldots,f(i)\} - (D \cup \{a\})$. □

Applying the lemma with $|D| = 1$ gives us the following theorem.

Theorem 2.25 MIN *is autoreducible.*

Proof. Fix a Gödel numbering φ. We will prove that given y and i, we can effectively decide whether $i \in \mathrm{TOT}_\varphi$ by making oracle queries to MIN_φ without querying y. Since $\mathrm{TOT}_\varphi \equiv_{\mathrm{T}} \mathrm{MIN}_\varphi$ this finishes the proof.

Consider two computable functions f_j $(j = 0, 1)$:

$$\varphi_{f_j(i)}(x) = \begin{cases} j & \text{if} \quad \varphi_i(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Let a_j $(j = 0, 1)$ be the minimal index of the function $(\lambda x)[j]$.

Let y and i be given. Using MIN_φ as oracle determine whether $\varphi_y(0)$ converges without making queries to y. (This is possible by the preceding lemma.) If so fix $j \in \{0,1\}$ such that $\varphi_y(0) \neq j$, else let $j = 0$. We can now compute the minimal index of $\varphi_{f_j(i)}$ without making queries to y using the same method as in theorem 2.11: compute initial segments (including undefined values) of functions φ_e with $e \in \mathrm{MIN}_\varphi \cap \{0, \ldots, f_j(i)\} - \{y\}$ until all but one of them is different from φ_y. This is possible, since $\varphi_{f_j(i)} \neq \varphi_y$ by choice of j, and we can decide $\varphi_e(x)\downarrow$ without querying y. Now $i \in \mathrm{TOT}_\varphi$ iff the index which is left is a_j. □

This result is nontrivial, since every degree above $\emptyset'$ contains a set which is not autoreducible (as proven by Jockusch and Paterson [21]).

2.6 Shortest Descriptions

Remember that we defined the set of shortest descriptions of a Gödel numbering φ to be

$$\mathrm{R}_\varphi = \{e : (\forall i < e)[\varphi_i(0) \neq \varphi_e(0)]\},$$

which is computable in $\emptyset'$. By this definition it is obvious that $\mathrm{R}_\varphi \subseteq \mathrm{MIN}_\varphi$. In particular R is also strongly effectively immune and ω-immune. Furthermore the construction showing that MIN is not hyperimmune works for R too.

We can conclude that $\emptyset'$ does not btt-reduce to R. A variation of the Meyer result gives us that $\emptyset'$ does wtt-reduce to R, and as in the case of MIN we can construct a Gödel numbering φ for which $\emptyset' \equiv_{\mathrm{tt}} \mathrm{R}_\varphi$. This, as in the case of MIN leaves us with the possibility that R is tt-equivalent to $\emptyset'$.

There has been related work in the area of Kolmogorov complexity. Let $C_\varphi(x) = \min\{\lg(e) : \varphi_e(0) = x\}$, where $\lg(x) = \lceil \log x \rceil$, the *Kolmogorov complexity of the number x (w.r.t. φ).* According to the textbook by Li and Vitányi, Kolmogorov knew that C_φ as a function of x is not computable, and they mention that C_φ is as hard as the halting problem (Exercise 2.7.7. attributed to Peter Gács). Using Arslanov's completeness criterion the result can be sharpened. We include a statement of the version of the criterion we will need.

Theorem 2.26 (Arslanov [21]) *If A is a c.e. set and $f \leq_{\mathrm{wtt}} A$ has no fixed-points, i.e., $\varphi_e \neq \varphi_{f(e)}$ for all e, then A is wtt-complete*[3].

Theorem 2.27 *Suppose A is an infinite c.e. set and f a computable partial function which agrees with C_φ on A. Then $\emptyset' \leq_{\mathrm{wtt}} f$. In particular any such function f has the same wtt-degree as $\emptyset'$.*

[3] Arslanov's criterion for *wtt*-reductions [21, Proposition III.8.17] is usually stated for functions which fulfill $W_e \neq W_{f(e)}$ for all e. Such a function can be transformed into one which is fixed-point free in our sense (see, for example, Exercise V.5.8 in Soare [28]).

Proof. Let B be an infinite computable subset of A, and f as described in the theorem. We will show how to compute a function g in f which is fixed-point free, i.e., $\varphi_e \neq \varphi_{g(e)}$. On input e search for $x \in B$ such that $f(x) > \lg e$ (and hence $C_\varphi(x) > \lg e$). Then search for some i for which $\varphi_i(0) = x$. Let $g(e) = i$. Since φ is a Gödel numbering we are sure to find such an i. Furthermore we know that we only have to check the first $e+1$ numbers in B beyond e to find an x as required. This gives us a *wtt*-reduction from g to f. □

With C_φ as a complexity function we can now give Kolmogorov's definition of randomness: x is *random (w.r.t.* φ*)* if its complexity C_φ is at least its length $\lg x = \lfloor \log_2 x \rfloor + 1$.

Definition 2.28 (Li and Vitányi [17]) *Define*

$$\mathrm{RAND}_\varphi = \{x : C_\varphi(x) \geq \lg x\},$$

the set of random strings *with regard to* φ*.*

Using Arslanov's completeness criterion again, one can show that $\emptyset' \leq_{\mathrm{wtt}} \mathrm{RAND}_\varphi$. Martin Kummer recently gave a surprising refinement of this result.

Theorem 2.29 (Kummer [14]) $\emptyset' \leq_{\mathrm{tt}} \mathrm{RAND}_\psi$ *for all Kolmogorov numberings* ψ*, but there is a Gödel numbering* φ*, for which* $\emptyset' \not\leq_{\mathrm{tt}} \mathrm{RAND}_\varphi$*.*

Kummer also mentions that a similar proof will show that there is a Gödel numbering φ for which the set $\{\langle x, e\rangle : (\exists i < e)[\varphi_i(0) = x]\}$ is not *tt*-complete. Although this comes closer to the set R of shortest descriptions as defined here, Kummer's methods do not seem applicable. Another result of Kummer's which does not carry over easily to R_φ is that RAND_φ is superterse.

3 Size-minimal Indices and Descriptions of Smallest Size

3.1 Size-minimal Indices

In the preceding sections we called an index minimal if it was the smallest index of a given function. In practice we might have different size measure than just the index itself. Most computer scientists for example would say the size of φ_i is $\lg i$, the length of the program, but other size measures have been considered too. There seem to be two reasonable requirements a size measure should meet: it should be computable, and there should only be finitely many indices of the same size. More formally a computable function s from ω to ω is called a *size function* if $s^{-1}(n) = \{m : s(m) = n\}$ is finite for all n. This definition might be found too restrictive in its insistence on the computability of s. We will return to this question in the section on $\subseteq$-minimal indices. For now we restrict s to be computable. Consider the following generalization of MIN.

Definition 3.1 (Bagchi [1]) *For a Gödel numbering* φ *and a size function* s *define*

$$\mathrm{MIN}_{\varphi,s} = \{e : (\forall i)[s(i) < s(e) \Rightarrow \varphi_i \neq \varphi_e]\},$$

the set of size-minimal indices of φ*. As usual we will drop* φ *if not needed. Dropping* s *means that* s *is the identity function.*

Let us first look at a special case: if a canonical index of $s^{-1}(n)$ can be computed effectively from n we call s a *strong size function*. In this case most of the results for MIN carry over to MIN_s, for example $\emptyset'' \equiv_{\mathrm{T}} \mathrm{MIN}_s$ (which was proved by Bagchi [1]). We will not pursue this question here, since the situation becomes much more interesting for general size functions.

A closer examination of MIN_s tells us that it lies in Σ^0_2 like MIN itself and that something slightly stronger is true.

Lemma 3.2 MIN_s *lies in* Σ^0_2 *uniformly in* s *(and* φ*).*

Proof. Note that $e \in \mathrm{MIN}_s$ if and only if

$$\begin{aligned}(\exists k)[\quad & (\forall i > k)[s(i) > s(e)] \wedge \\ & (\forall i < k)(\exists x)[s(i) < s(e) \Rightarrow \varphi_e(x) \neq \varphi_i(x)]].\end{aligned}$$

The $(\forall i < k)(\exists x)$ can be made part of the first existential quantifier. Then both $(\forall i > k)[s(i) > s(e)]$ and $\varphi_e(x) \neq \varphi_i(x)$ are decidable with a $\emptyset'$ oracle uniformly in s (and φ), even if s is not total. □

Bagchi [1] proved that $\emptyset'' \equiv_{\mathrm{T}} \mathrm{MIN}_s \oplus \emptyset'$, but he leaves unanswered the question of whether $\emptyset'$ reduces to MIN_s. We know already that if s is the identity function then we can make $\mathrm{MIN}_{\varphi,s}$ tt-complete for some Gödel numbering φ. It is still open whether $\emptyset'$ Turing reduces to MIN_s, but we have the following result which shows that if such a reduction exists it has to be a proper Turing reduction, and not a wtt-reduction. In this respect it is interesting to compare this to the result on shortest descriptions in the next section.

Theorem 3.3 *There is a computable size function* s *(independent of the Gödel numbering) such that* $\emptyset'$ *does not wtt-reduce to* MIN_s*.*

The theorem is a consequence of a new result which is given below and the classical result by Friedberg and Rogers that $\emptyset'$ does not wtt-reduce to a hyperimmune set [28, Exercise 2.16].

Theorem 3.4 *There is a computable size function* s *(independent of the Gödel numbering) such that* MIN_s *is hyperimmune.*

Proof. We will in fact construct a computable size function s such that $\mathrm{MIN}_{\psi,s}$ is hyperimmune for every effective numbering ψ. Let $\Psi(z, e, x)$ be a universal function. Then $(\psi^z_e)_{e\in\omega} := (\Psi(z, e, \cdot))_{e\in\omega}$ will contain all effective numberings, and in particular all Gödel numberings as z ranges over ω.

Fix a particular Gödel numbering φ. The construction will be a straight-forward priority argument fulfilling the requirements

$$R_{e,z} : \text{if } (D_{\varphi_e(x)})_{x\in\omega} \text{ is a strong disjoint array, then there is an } x \text{ for which } D_{\varphi_e(x)} \cap \mathrm{MIN}_{\psi^z,s} = \emptyset,$$

for all $e, z \in \omega$.

Stage $t = 0$. Initially s is undefined on all values.

Stage t. If $s(t)$ is still undefined at this stage, then define it to have value t. We say $\langle e, z\rangle \leq t$ *requires attention* at stage t if $R_{e,z}$ has not received attention yet, and there is a $w \leq t$ such that

- $\varphi_{e,t}(w)$ is defined, and
- $D_{\varphi_e(w)}$ only contains elements whose size is defined and is at least $\langle e, z\rangle + 1$.

Let $\langle e, z\rangle$ be the minimal element that requires attention (if any) and fix the corresponding w. We say that $\langle e, z\rangle$ *receives attention*. Let $D = D_{\varphi_e(w)}$. Effectively find a new (finite) set E of ψ^z indices of the functions indexed by D such that the elements of E have not been assigned a length yet. Namely $\{\psi^z_i : i \in D\} = \{\psi^z_i : i \in E\}$. Assign a size of $\langle e, z\rangle$ to each element in E.

By construction s is a computable, total function, and it is a size function since each requirement $R_{e,z}$ assigns the value $\langle e, z\rangle$ to a finite number of functions.

Note that a requirement that receives attention is met immediately and never injured afterwards. Now suppose not all requirements are fulfilled, and let $\langle e, z\rangle$ be minimal such that $R_{\langle e,z\rangle}$ is not met. We can choose a stage $t' > \langle e, z\rangle$ after which no R_j with $j < \langle e, z\rangle$ acts. Then the sizes assigned from stage t' on will be at least $\langle e, z\rangle + 1$. Since $(D_{\varphi_e(x)})_{x\in\omega}$ is a strong disjoint array (otherwise $R_{\langle e,z\rangle}$ would be fulfilled) there is a w for which $D_{\varphi_e(w)}$ contains only elements of size at least $\langle e,z\rangle+1$ forcing $R_{\langle e,z\rangle}$ to act and become fulfilled. □

Note though that MIN_s is not hyperhyperimmune (as Bagchi observes).

Remark. If we have a function f which on input e returns a size-minimal index of φ, i.e., $f(e)\in\mathrm{MIN}_s$ and $\varphi_e=\varphi_{f(e)}$ (there might be many such functions), then it is easy to see (using the same tricks as for minimal indices) that $\emptyset''\le_{\mathrm{T}} f$, and $f\le_{\mathrm{T}}\mathrm{MIN}_s\oplus\emptyset'$.

On the other hand it is easy to see (using the coding techniques with which MIN was made tt-complete) that there is a computable size function s (independent of the Gödel numbering) such that MIN_s is tt-complete for $\emptyset''$.

The set MIN_s shares some immunity properties with MIN. For example the proof of ω-immunity of MIN can be easily adapted.

Theorem 3.5 MIN_s *is ω-immune (for all computable size functions s).*

Corollary 3.6 $\emptyset'\not\le_{\mathrm{btt}}\mathrm{MIN}_s$ *(for any computable size function s).*

Although we do not expect it to be either effectively or even strongly effectively immune in general (and it is easy to construct s where it is neither), it is constructively immune. We will prove a more general result which is based on a proof by Owings [22].

Definition 3.7 *A set A is* constructively immune *if it is infinite and there is a computable partial function ψ such that if W_e is infinite, then $\psi(e)\downarrow$ and $\psi(e)\in W_e-A$.*

Let $I(e)=\{i:\varphi_i=\varphi_e\}$, the set of indices of φ_e.

Definition 3.8 *We call $H:\mathcal{P}(\omega)\to\mathcal{P}(\omega)$ an* effective choice functional *if there is a computable g such that*

- *$H(I(e))\subseteq I(e)$,*
- *$H(I(e))\cap W_{g(e)}=\emptyset$, and*
- *$W_{g(e)}$ cofinite,*

for all e.

Effective choice functionals should not be confused with Owings [22] effective choice functions which are computable partial functions H from $\mathcal{P}(\omega)$ to ω such that $H(A)\in A$ for all $A\in\mathrm{dom}(H)$.

Theorem 3.9 *Suppose that for a set M there is a computable function g such that $I(e)\cap M\cap W_{g(e)}=\emptyset$ and $W_{g(e)}$ is cofinite for all e. Then M is constructively immune.*

Proof. Using the Recursion Theorem define a computable function f fulfilling:

$$\varphi_{f(e)}(x)=\begin{cases}\varphi_i(x) & \text{if } \langle i,t\rangle \text{ is the smallest } \langle i',t'\rangle \text{ with } i'\in W_{e,t'}\cap W_{g(f(e)),t'},\\ \uparrow & \text{otherwise}\end{cases}$$

and let $\psi(e)$ be the i for which $\langle i,t\rangle$ is the smallest $\langle i',t'\rangle$ with $i'\in W_{e,t'}\cap W_{g(f(e)),t'}$. Assume that W_e is infinite. This implies that $W_e\cap W_{g(f(e))}$ is infinite, and $\varphi_{f(e)}=\varphi_{\psi(e)}$. Thus we conclude $\psi(e)\in I(f(e))$, but $\psi(e)\in W_{g(f(e))}$ and hence $\psi(e)\notin I(f(e))\cap M$ and therefore $\psi(e)\notin M$. □

Corollary 3.10 *If H is an effective choice functional, then $M=\{H(I(e)):e\in\omega\}$ is constructively immune.*

Proof. For an effective choice functional we have $M\cap I(e)\cap W_{g(e)}=H(I(e))\cap W_{g(e)}=\emptyset$. Hence we can apply the theorem. □

Corollary 3.11 MIN_s *is constructively immune for every computable size function s.*

Proof. Let $H(I(e)) = \{i \in I(e) : (\forall j \in I(e))[s(j) \geq s(i)]\}$. We define g computable such that $W_{g(e)} = \{i : s(i) > s(e)\}$. Then $W_{g(e)}$ is cofinite since s is a size function, and it witnesses that H is an effective choice functional. Now the corollary applies, since $\mathrm{MIN}_s = \{H(I(e)) : e \in \omega\}$. □

Note that, for example, the same result holds for the ith-minimal indices.

The next result is immediate from Owings' paper.

Lemma 3.12 (Owings [22]) *Suppose that H is a computable partial functional whose domain includes all infinite subsets of ω, and such that $H(A)$ is a finite subset of A for all A in the domain of H. Then H is an effective choice functional.*

Hence we can draw the following conclusion.

Corollary 3.13 *Suppose that H is a computable partial functional whose domain includes all infinite subsets of ω, and such that $H(A)$ is a finite subset of A for all A in the domain of H. Then $M = \{H(I(e)) : e \in \omega\}$ is constructively immune.*

In Owings paper it is shown that M is effectively immune under the assumptions of the corollary. Xiang [30] showed that the notions of constructive and effective immunity are independent.

3.2 Descriptions of smallest size

Consider the set $R_s = \{e : (\forall i)[s(i) < s(e) \Rightarrow \varphi_i(0) \neq \varphi_e(0)]\}$. (As usual φ remains in the background.)

Since $R_s \subseteq \mathrm{MIN}_s$ the immunity properties carry over from MIN_s to R_s.

Corollary 3.14

- *R_s is ω-immune (for every computable size function s),*
- *There is a computable size function s such that R_s is hyperimmune,*
- *R_s is constructively immune.*

Hence we also know that $\emptyset' \not\leq_{\mathrm{btt}} R_s$ (for any computable size function s). Concerning the degree of R_s for once we can get a tight result. By the second item of the corollary there is a Gödel numbering for which $\emptyset' \not\leq_{\mathrm{wtt}} R_s$. Since R_s is a 2-c.e. (d.c.e.) set in which we can compute a fixed-point free function we can conclude $\emptyset' \leq_{\mathrm{T}} \mathrm{R}_s$ using the generalized Arslanov completeness criterion [10].

Proposition 3.15 $\emptyset' \equiv_{\mathrm{T}} \mathrm{R}_s$ *(for all computable size functions s).*

The proposition leaves us with a slightly unsatisfactory situation: we know that a Turing reduction exists, but we cannot explicitly present it. Part of the reason is that the generalized Arslanov completeness criterion is nonuniform. It does not yield a Turing reduction uniformly in s (and as a matter of fact it cannot, even for d.c.e. sets [10, Theorem 6.4]). The challenge remains to show that the uniform analogue of the proposition is false, or to exhibit a direct reduction from $\emptyset'$ to R_s which is uniform.

4 Minimal indices of total, finite and infinite functions

Several natural variants of MIN result from restricting our attention to certain classes of functions, like total, infinite, or finite functions. Thus we might consider $\mathrm{MIN}^{\mathrm{fin}} = \mathrm{MIN} \cap \mathrm{FIN}$, $\mathrm{MIN}^{\mathrm{tot}} = \mathrm{MIN} \cap \mathrm{TOT}$ or $\mathrm{MIN}^{\mathrm{inf}} = \mathrm{MIN} \cap \mathrm{INF}$ the minimal indices of finite, total and infinite functions, respectively. Whereas a standard proof shows that $\emptyset'' \equiv_{\mathrm{T}} \mathrm{MIN}^{\mathrm{fin}}$, it can be proved that $\mathrm{MIN}^{\mathrm{tot}}$ and $\mathrm{MIN}^{\mathrm{inf}}$ are wtt-complete for $\emptyset''$ rather than just Turing complete. The basic trick for this result is due to Lance Fortnow. It will also serve us well in the next section on $=^*$-minimal indices.

Theorem 4.1 (Fortnow (personal communication)) *$\emptyset'' \leq_{\rm wtt} {\rm MIN}^{\rm tot}$.*

Proof. We skip the proof that $\emptyset' \leq_{\rm wtt} {\rm MIN}^{\rm tot}$ which follows lines familiar from MIN. Define a computable function f as follows:

$$\varphi_{f(e)}(x) = \begin{cases} (\mu s)[\varphi_{e,s}(x)\downarrow] & \text{if } \varphi_e(y)\downarrow, \\ \uparrow & \text{otherwise,} \end{cases}$$

Then $e \in {\rm TOT}$ if and only if there is an $i \in \{0,\ldots,f(e)\} \cap {\rm MIN}^{\rm tot}$ such that $\varphi_{e,\varphi_i(x)}(x)\downarrow$ for all $x \in \omega$. Since $i \in {\rm TOT}$ the last condition can be decided in $\emptyset'$ which *wtt*-reduces to ${\rm MIN}^{\rm tot}$. Since the queries can furthermore be bounded effectively this yields a *wtt*-reduction from $\emptyset''$ to ${\rm MIN}^{\rm tot}$. □

Some slight adjustments will also give $\emptyset'' \leq_{\rm wtt} {\rm MIN}^{\rm inf}$. These two results are particularly interesting in the light of the observation that $\emptyset' \leq_{\rm tt} B$ and $A \leq_{\rm wtt} B$ imply $A \leq_{\rm tt} B$ (thanks to Martin Kummer for pointing this out [21, Proof of Proposition VI.5.8]). Hence if we could show that $\emptyset' \leq_{\rm tt} {\rm MIN}^{\rm tot}$ we would already know that $\emptyset'' \leq_{\rm tt} {\rm MIN}^{\rm tot}$ (and the same for ${\rm MIN}^{\rm inf}$).

5 $=^*$-minimal indices

The $=^*$-minimal indices are yet another very interesting and strange variant of MIN, first defined by John Case. Remember that two partial functions f and g are said to be almost always equal (written as $f =^* g$) if they agree on all but finitely many inputs.

Definition 5.1 (Case [3]) *For a Gödel numbering φ define*

$${\rm MIN}^*_\varphi = \{e : (\forall i < e)[\varphi_i \neq^* \varphi_e]\},$$

the $=^*$-minimal indices of φ.

As regards dropping φ the same conventions that were used for MIN apply. Let us first note some facts.

Lemma 5.2 *(i)* (Case [3]) ${\rm MIN}^*$ *is Σ^0_2-immune, i.e., it has no infinite subset in Σ^0_2.*

(ii) ${\rm MIN}^* \in \Pi^0_3 - \Sigma^0_2$.

(iii) There is a Kolmogorov numbering φ such that ${\rm MIN}^*_\varphi$ is d-complete for Π^0_3.

(iv) ${\rm MIN}^*$ *is k-immune for every k, hence $\emptyset'$ does not btt-reduce to ${\rm MIN}^*$.*

(v) ${\rm MIN}^*$ *is not hyperimmune.*

Proof. For *(i)* use a result by Arslanov, Nadirov, and Solov'ev on almost fixed points [10, Theorem 2.1, Lemma 4.1]. Then *(ii)*: ${\rm MIN}^* \notin \Sigma^0_2$ is an immediate consequence and ${\rm MIN}^* \in \Pi^0_3$ is easily checked. Using the Kolmogorov numbering φ from Theorem 2.17 yields *(iii)* (even using the same reduction). For *(iv)* note that ${\rm MIN}^* \subset {\rm MIN}$. For *(v)* the same proof as for MIN works. □

What about the degree of ${\rm MIN}^*$? The question seems to be more difficult than for MIN, since reducing $\emptyset'$ to ${\rm MIN}^*$ poses serious problems. Since we only have $=^*$-minimal indices, all the information a computable algorithm can find about φ could be in the initial faulty part. Therefore the ideas used for MIN do not work here. The best we can prove is the following theorem.

Theorem 5.3 *${\rm MIN}^* \oplus \emptyset' \equiv_{\rm T} \emptyset'''$.*

The heart of the proof is the following lemma.

Lemma 5.4 *${\rm MIN}^* \oplus \emptyset' \geq_{\rm T} \emptyset''$.*

Proof. Fix a Gödel numbering φ. We will show how to enumerate TOT_φ in $\mathrm{MIN}^* \oplus \emptyset'$. Since obviously $\overline{\mathrm{TOT}_\varphi}$ is c.e. in $\emptyset'$ this proves the theorem. Define two computable functions f and g as follows:

$$\varphi_{f(e)}(x) = \begin{cases} (\mu s)[(\forall y \le x)[\varphi_{e,s}(y)\downarrow]] & \text{if } (\forall y \le x)[\varphi_e(y)\downarrow], \\ \uparrow & \text{otherwise,} \end{cases}$$

and

$$\varphi_{g(e)}(x) = \begin{cases} (\mu\langle y,s,z\rangle)[y \ge x \wedge \varphi_{e,s}(y)\downarrow = z]] & \text{if there is a } y \ge x \text{ such that } \varphi_e(y)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Fix a minimal such that $\varphi_a =^* (\lambda x)[\uparrow]$, i.e., a is the $=^*$-minimal index of the everywhere undefined function. Consider the following algorithm.

On input e search for $i \le f(e)$ with $i \in \mathrm{MIN}^*$ and $i \ne a$ and n such that $(\forall x)[\varphi_{e,\max\{n,\varphi_{g(i)}(x)\}}(x)\downarrow]$ is true. Halt if such i and n are found.

First note that if $i \in \mathrm{MIN}^*$, and $i \ne a$, then φ_i is defined infinitely often, hence $\varphi_{g(i)}$ is total, i.e., $\varphi_{e,\max\{n,\varphi_{g(i)}(x)\}}(x)\downarrow$ can be effectively decided in e, i and x, hence $(\forall x)[\varphi_{e,\max\{n,\varphi_{g(i)}(x)\}}(x)\downarrow]$ can be decided by the $\emptyset'$ oracle. Hence the algorithm will work with a $\mathrm{MIN}^* \oplus \emptyset'$ oracle. If the algorithm terminates, then in particular $\varphi_e(x)\downarrow$ for all x, and so $e \in \mathrm{TOT}$. We only have to argue that the algorithm does terminate for $e \in \mathrm{TOT}$. Suppose $e \in \mathrm{TOT}$. Then $f(e)$ is an index of a total function which for every x gives an upper bound on the number of steps it takes φ_e to converge on x: $\varphi_{e,\varphi_{f(e)}}(x)\downarrow$ for all x. This function has a $=^*$-minimal index $i \in \mathrm{MIN}^*$ such that $\varphi_{f(e)}(x) = \varphi_i(x)$ for all $x \ge n$. Now $\varphi_{g(i)}(x) \ge \varphi_i(x)$ for all x, and hence $\varphi_{e,\max\{n,\varphi_{g(i)}(x)\}}(x)\downarrow$ for all x, so the algorithm terminates. □

Proof of Theorem 5.3. Since $\mathrm{MIN}^* \in \Pi^0_3$ it is clear that $\mathrm{MIN}^* \oplus \emptyset' \le_\mathrm{T} \emptyset'''$. On the other hand we just proved that $\mathrm{MIN}^* \oplus \emptyset' \ge_\mathrm{T} \emptyset''$. Thus it will be sufficient to show that $\emptyset''' \le_\mathrm{T} \mathrm{MIN}^* \oplus \emptyset''$.

By Lemma 5.2 there is a Gödel numbering φ such that $\emptyset''' \equiv_\mathrm{T} \mathrm{MIN}^*_\varphi$, so we are done if we can prove that $\mathrm{MIN}^*_\varphi \le_\mathrm{T} \mathrm{MIN}^*_\psi \oplus \emptyset''$ for every Gödel numbering ψ. Fix ψ and a computable translation function f from φ to ψ, i.e., $\psi_{f(x)} = \varphi_x$ for all x. We need an algorithm to decide whether $e \in \mathrm{MIN}^*_\varphi$. Let $m = \max\{f(i) : i \le e\}$, and $I = \mathrm{MIN}^*_\psi \cap \{0,1,\ldots,m\}$. Then I contains $=^*$ minimal indices (w.r.t. ψ) for $\varphi_0,\ldots,\varphi_e$. For each φ_i we can find $i' \in I$ such that $\varphi_i =^* \varphi_{i'}$ using the $\emptyset''$ oracle (note that we do not have to decide $\varphi_i =^* \varphi_j$ to do that, that would be a Σ^0_3 complete task). Now $e \in \mathrm{MIN}^*_\varphi$ if and only if no $i < e$ has the same $=^*$-minimal index (w.r.t. ψ) as e. □

Although the theorem does not allow us to pin down the degree of MIN^* exactly we can at least conclude that MIN^* does not lie in Σ^0_3.

Corollary 5.5 $\mathrm{MIN}^* \in \Pi^0_3 - \Sigma^0_3$.

6 $\subseteq$-minimal programs, decision tables and noncomputable size measures

We mentioned earlier that it might be natural to consider noncomputable size measures. Call a function $s : S \to \omega$ a *weak size function* if $s^{-1}(n)$ is finite for all n, where $S \subseteq \omega$ is a set of indices we are interested in. Pager [24, 25] and Young [29] suggested some examples of weak size functions which are based on Blum's complexity measures. In these examples the complexity of a function becomes part of its size.

Let us look at an example which measures the size of finite functions.

$$s(e) = \begin{cases} t+e & \text{if } t \text{ is minimal such that } \mathrm{dom}(\varphi_e) \subseteq \{0,\ldots,t\} \\ & \text{and } \varphi_{e,t}(x)\downarrow \text{ for all } x \in \mathrm{dom}(\varphi_e) \\ \uparrow & \text{otherwise} \end{cases}$$

Then s is a weak size function (computable in $\emptyset'$) which tells us the domain of finite functions and how long it takes to compute the values in the domain. This seems to be a natural size function taking into account both time and space. It is essentially the one suggested by Young [29]; Pager [25] later generalized it to include infinite functions. For this particular weak size function we do not need the full power of $\emptyset''$ to compute a minimal index for every finite function, since s gives us an upper bound on both the running time and the index of a size-minimal index. More formally there is a partial function f computable in $\emptyset'$ such that $\varphi_e = \varphi_{f(e)}$ and $f(e) \in \mathrm{MIN}_s$ whenever φ_e is a finite function.

Remark. In the light of our newfound interest in weak size function let us return to size-minimal indices and see what happens there. Bagchi [1] proved that $\emptyset'' \leq_{\mathrm{T}} \mathrm{MIN}_s \oplus \emptyset'$ for size functions. For weak size functions this becomes $\emptyset'' \leq_{\mathrm{T}} \mathrm{MIN}_s \oplus s'$, where s' is the jump of the graph of s. This means that MIN_s and s cannot be computationally easy at the same time. Bagchi observed that s can be chosen in such a way (using the Friedberg numbering) that MIN_s becomes computable. It might be interesting to note that in the other direction Kummer [12] used minimal indices to give an easy priority-free proof of the existence of a Friedberg numbering.

For the rest of this section we will concentrate on weak size functions. The variant of minimal indices we look at next is suggested by the work of David Pager [23, 24]. For two partial functions f and g we write $f \subseteq g$ if $f(x) = g(x)$ for all x in the domain of f.

Definition 6.1 (Bagchi [1]) *Let*

$$\mathrm{MIN}^{\subseteq}_{\varphi,\mathrm{s}} = \{e : (\forall i)[s(i) < s(e) \Rightarrow \varphi_e \not\subseteq \varphi_i]\},$$

the set of $\subseteq$-size-minimal indices of φ.

The idea behind the definition of $\mathrm{MIN}^{\subseteq}_{\mathrm{s}}$ is that if you are looking for a minimal index of a function that computes f you might not care what happens outside the domain of f. In this case some function $g \supseteq f$ might have a much smaller minimal index than f itself.

It is easy to see that $\mathrm{MIN}^{\subseteq}_{\mathrm{s}} \leq_{\mathrm{T}} \emptyset'' \oplus s'$ (s' allows you to compute all indices of a given size and then the $\emptyset''$ oracle will do the rest). As for size-minimal indices it can be proved that $\emptyset'' \leq_{\mathrm{T}} \mathrm{MIN}^{\subseteq}_{\mathrm{s}} \oplus s'$.

How difficult is it to compute $\subseteq$-size-minimal indices of certain classes of functions? Consider, for example, a function f that computes a $\subseteq$-size-minimal index of every computable partial function given by its index. That is, f has to fulfill $\varphi_e = \varphi_{f(e)}$ and $f(e) \in \mathrm{MIN}^{\subseteq}_{\mathrm{s}}$. Then one easily shows that $\emptyset'' \leq_{\mathrm{T}} f$.

We conclude that although the complexity of $\mathrm{MIN}^{\subseteq}_{\mathrm{s}}$ itself might fluctuate with s, the problem of finding a $\subseteq$-size-minimal program is never easy. Remember that the same is true of MIN_s.

At the beginning of this section we discussed a particular weak size function for which it was computable in $\emptyset'$ to find size-minimal indices for finite functions. What happens if we ask for $\subseteq$-*size minimal indices* of finite functions? Let us refine the problem. The restriction to finite functions suggests a different representation from the one used so far. Instead of representing the function by its index, we specify its behavior as a set of pairs: $\{(x_i, b_i) : b_i \in \{0,1\} \text{ and } 1 \leq i \leq n\}$. This means the value of the function has to be b_i on input x_i (for all $1 \leq i \leq n$). Note that we limit ourselves to $\{0,1\}$-valued functions. If we represent functions in this way we call them *decision tables*. Consequently the (x_i, b_i) are called the *entries of the decision table*.

The question of how difficult it is to find a $\subseteq$-size-minimal index for a decision table was first investigated by David Pager [24, 25]. He proved that even if we restrict ourselves to decision tables with just two entries, the problem is undecidable, regardless of the complexity of the size function. Let $\zeta(x, y)$ denote the function with the decision table $\{(x, 0), (y, 1)\}$.

Theorem 6.2 (Pager [25]) *For every weak size function s there is a constant c such that no total computable function f fulfills*

- $\zeta(c, x) \subseteq \varphi_{f(c,x)}$,
- $\zeta(c, x) \not\subseteq \varphi_e$ *for all e with* $s(e) < s(f(c, x))$

Using his proof and improving it slightly we can show that computing a $\subseteq$-size-minimal index of a two-entry decision table has at least the complexity of the halting problem.

Theorem 6.3 *If s is a weak size function and f a total function which computes $\subseteq$-size-minimal indices of two entry decision tables, i.e.,*

- $\zeta(x, y) \subseteq \varphi_{f(x,y)}$,
- $\zeta(x, y) \not\subseteq \varphi_e$ *for all e with* $s(e) < s(f(x, y))$,

then $\emptyset' \leq_{\mathrm{T}} f$.

In the light of earlier results the last theorem and Pager's original result seem surprising, since they only require s to be finite-one. All the tricks we have seen so far (using the recursion theorem) are no longer applicable but have to be substituted by a more involved argument.

Proof. Assume that f fulfills the hypothesis of the theorem. Let M be a Turing complete maximal set (Friedberg's set will do since it is effectively maximal). Split M into two c.e., noncomputable sets S and T, i.e., $M = S \cup T$ and $S \cap T = \emptyset$. In this case S and T are *strongly inseparable*, namely if U and V are c.e. sets such that $S \subseteq U \subseteq \overline{V} \subseteq \overline{T}$, then $S =^* U$ and $T =^* V$ (the easy proof can be found in a paper by Cleave [5]). Define

$$g(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{if } x \in T \\ \uparrow & \text{otherwise.} \end{cases}$$

Fix an index a of g and let I_1 be the finite set of indices which have at most the size of a, namely $H = \{i : s(i) \leq s(a)\}$. From the indices in H we want to filter out those which belong to a function that is 1 infinitely often outside of M. To this purpose define

$$I = \{i \in I_1 : |\overline{M} \cap \{x : \varphi_i(x) = 1\}| = \infty\}.$$

Let $X = \bigcap_{i \in I} \{x : \varphi_i(x) = 1\}$ (note that if I is empty, then $X = \omega$). For all $i \in I$ we have that $\overline{M} \cap \overline{\{x : \varphi_i(x) = 1\}}$ is a finite set (M is maximal), and since I is finite, the set $\overline{M} \cap \overline{X} = \bigcup_{i \in I} \overline{M} \cap \overline{\{x : \varphi_i(x) = 1\}}$ is finite. Since $\overline{M}$ is infinite this implies that $\overline{M} \cap X$ is infinite.

We claim that $S \cap X$ is infinite. If $S \cap X$ was finite, then $V = (X - S) \cup T$ would be a c.e. set which separates S and T, but for which $V =^* T$ is false, contradicting the strong inseparability of S and T. Hence we can choose an element $c \in S \cap X$. Note that if $x \in T$, then $f(c, x) \in H$ (since H contains an index of g). On the other hand if $x \in \overline{M}$, then $f(c, x) \notin H$ for almost all x by choice of c.

Repeating the argument with the set $I' = \{i \in H : |\overline{M} \cap \{x : \varphi_i(x) = 0\}| = \infty\}$, we get a constant d such that if $x \in S$, then $f(x, d) \in H$, and if $x \in \overline{M}$, then $f(x, d) \notin H$ for almost all x by choice of d. Since M is the union of S and T this means that $x \in M$ if and only if $f(c, x) \in H$ or $f(x, d) \in H$ for almost all x. Since M was chosen to be Turing complete (and H is finite) this finishes the proof. □

Remark. It is easy to see that a function f as in the theorem can be computed with an oracle for s', the jump of the size function. In case s is computable that implies that the complexity of f is exactly the complexity of the halting problem.

Corollary 6.4 *Finding a $\subseteq$-size-minimal index for a decision table with two or more entries is at least as difficult as solving the halting problem.*

Finding $\subseteq$-size-minimal indices for decision tables with one entry can be computable or not, depending on the size function s, as observed by Pager.

7 Learning Theory

The last variant of MIN we want to mention is the one considered in learning theory. Instead of insisting on the exact minimal index of a function we allow some freedom. Remember that we defined $\min_\varphi(i)$ to be the minimal φ-index of the function φ_i.

Definition 7.1 *For a computable function h with $h(x) > x$ for all x let*

$$h\text{-MIN}_\varphi = \{e : e < h(\min_\varphi(e))\}.$$

It is easy to see that given h and φ we can construct a Gödel numbering ψ by stretching φ out using h such that $h\text{-MIN}_\varphi =^* \text{MIN}_\psi$. That means that all results true for MIN (i.e., for MIN_ψ for *all* ψ) which are robust under finite variations are automatically true for h-MIN. Also results that are not affected by the stretching out (like the construction of a Gödel numbering φ for which MIN_φ is d-complete for Σ_2^0) can be carried over to the case of nearly-minimal indices.

Learning theory investigates how difficult it is to learn a function by returning an index in h-MIN. There are also variants which instead of trying to approximate $\min_\varphi(e)$ allow more freedom by accepting a fixed (or finite) number of errors in the function (rather like MIN^*). A short history of this area can be found in a paper by Case, Jain and Suraj [4]. But that's another story and shall be told another time.

8 Open Questions

Several interesting questions remain open. Foremost is Meyer's original question whether all MIN_φ are tt-equivalent. Because of the construction of a Gödel numbering φ for which MIN_φ is tt-complete, this would imply that all MIN_φ belong to the tt-degree of $\emptyset''$.

Knowing that $\emptyset' \not\leq_{\text{btt}}$ MIN and $\emptyset' \leq_{\text{wtt}}$ MIN leaves us with the tantalizing question of whether $\emptyset' \leq_{\text{tt}}$ MIN. If this should indeed be the case, and it could be proved that $\emptyset'' \leq_{\text{wtt}}$ MIN, we would already have $\emptyset'' \leq_{\text{tt}}$ MIN, by a general result (use the tt-reduction of $\emptyset'$ to figure out whether the wtt-reduction will converge). In this respect it is worth remembering that some variations of MIN like MIN^{tot} and MIN^{inf} are wtt-complete for Σ_2^0 (and not only Turing complete).

We would like to settle the degree of MIN^* by either showing that that $\emptyset' \leq_{\text{T}} \text{MIN}^*$ or by constructing a Gödel numbering for which this is not the case. Even constructing a Gödel numbering φ such that $\emptyset' \not\leq_{\text{tt}} \text{MIN}^*$ would be interesting, since it might carry over to MIN and answer Meyer's open question.

We know that MIN_φ is not approximable for some Gödel numbering φ. This make this particular MIN_φ a natural example of an autoreducible, non-approximable set (Kummer and Stephan [15] showed that approximable sets are autoreducible). If we could prove that MIN is not approximable the example would be even more convincing. If, on the other hand, there is a Gödel numbering φ for which MIN_φ is approximable, this would prove that not all MIN_φ are btt-equivalent (because approximable sets are closed downwards under *btt*-reductions). A stronger result is already known (Kinber's theorem), but the proof might be easier.

The big open question for size-minimal indices is whether MIN_s is Turing-complete for $\emptyset''$, but there are also a host of other questions which have not been asked yet: is MIN_s autoreducible, approximable, superterse, etc. Similarly it is an open problem to determine what the possible degrees of $\mathrm{MIN}_s^{\subseteq}$ are.

One approach to MIN and MIN_s is through their $\emptyset'$-versions: shortest descriptions and descriptions of smallest size. The hope is that some of the open questions might be easier to answer when asked about R and R_s, but at the same time might yield an insight on how to approach the original problem. It seems, however, that we do not understand the relationship between R_s, RAND_φ and MIN_s very well.

Acknowledgments Thanks go to Martin Kummer [13] for giving me an extensive overview of the history of the MIN-problem, Bill Gasarch for suggesting [22], and Lance Fortnow for help with some of the results. Further thanks to Steve Fenner for long discussions on bounded reducibilities which eventually made it into a separate paper [6]. Several helpful remarks came from an anonymous referee who in particular deserves credit for pointing out the possibility of a short proof of Theorem 2.10 similar to the one presented here (using the recursion theorem instead of inseparable sets). Finally I would wish to thank Sophie Laplante and Amber Settle for diligent proofreading.

References

[1] Amitava Bagchi. Economy of Descriptions and Minimal Indices. MAC Technical Memorandum, MIT, 1972.

[2] Manuel Blum. On the size of machines. *Information and Control*, 11, 257–265, 1967.

[3] John Case. *email communication*, 1996.

[4] John Case, S. Jain, M. Suraj. Not-So-Nearly-Minimal-Size Program Inference. *Lecture Notes in Artificial Intelligence*, 961, 77–96, 1995.

[5] J. P. Cleave. Some Properties of Recursively Inseparable Sets. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 16, 187–200, 1970.

[6] Stephen Fenner, Marcus Schaefer. Bounded Immunity and Btt-Reductions. *Unpublished manuscript*, 1997; a preliminary version appeared as: A note on a variant of immunity, btt-reducibility, and minimal indices. *Technical Report TR96-8-13, University of Southern Maine*, `http://www.cs.usm.maine.edu/report/TR96-8-13.ps`, 1996.

[7] R.M. Friedberg. Three theorems on recursive enumeration. *Journal of Symbolic Logic*, 23, 309–316, 1958.

[8] Carl G. Jockusch. Semirecursive sets and positive reducibility. *Trans. Amer. Math. Soc.*, 131, 420–436, 1968.

[9] Carl G. Jockusch. Weakly Semirecursive Sets. *The Journal of Symbolic Logic*, 55, 2, 637–644, 1990.

[10] Carl G. Jockusch, Jr., M. Lerman, R. I. Soare, R. M. Solovay. Recursively enumerable sets modulo iterated jumps and extensions of Arslanov's completeness criterion. *Journal of Symbolic Logic*, 43,4, 1288–1323, 1989.

[11] Jefim Kinber. On btt-degrees of sets of minimal numbers in Gödel numberings. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 23, 201–212, 1977.

[12] Martin Kummer. An easy priority-free proof of a theorem of Friedberg. *Theoretical Computer Science*, 74, 249–251, 1990.

[13] Martin Kummer. *email communication*, March, 1996.

[14] Martin Kummer. On the Complexity of Random Strings. *12th Annual Symposium on Theoretical Aspects of Computer Science*, 1046, 25–36, 1996.

[15] Martin Kummer, Frank Stephan. Some Aspects of Frequency Computation. *Technical Report 21/91*, Universität Karlsruhe, October, 1991.

[16] Martin Kummer, Frank Stephan. Recursion Theoretic Properties of Frequency Computation and Bounded Queries. *Information and Computation* 120(1), 59–77, 1995.

[17] Ming Li, Paul Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications.* Springer Verlag, New York, Second Edition, 1997.

[18] G. B. Marandžjan. On the sets of minimal indices of partial recursive functions. *Mathematical Foundations of Computer Science 1979*, Lecture Notes in Computer Science 74, 372–374.

[19] G. B. Marandžjan. Selected Topics in Recursive Function Theory. *Technical Report 1990-75, August 1990*, Department of Computer Science, Technical University of Denmark.

[20] Albert R. Meyer. Program Size in Restricted Programming Languages. *Information and Control*, 21, 382–394, 1972.

[21] Piergiorgio Odifreddi. *Classical recursion theory.* North-Holland, Amsterdam, 1989.

[22] James C. Owings. Effective Choice Functions and Index Sets. *Journal of Computer and System Sciences*, 32, 370–373, 1986.

[23] David Pager. On the Problem of Finding Minimal Programs for Tables. *Information and Control*, 14, 550–554, 1969.

[24] David Pager. Further Results on the Problem of Finding Minimal Length Programs for Decision Tables. *Journal of the ACM*, 21, 2, 207–212, 1974.

[25] David Pager. On the Efficiency of Algorithms *Journal of the ACM*, 17, 4, 703–714, 1979.

[26] C. P. Schnorr. Optimal Enumerations and Optimal Gödel Numberings. *Mathematical Systems Theory*, 8, 182–191, 1974.

[27] Robert I. Soare. Computability and Recursiveness. *Bulletin of Symbolic Logic*, 3, 284–321, 1996.

[28] Robert I. Soare. *Recursively Enumerable Sets and Degrees.* Springer, New York, 1987.

[29] Paul D. Young. A Note on "Axioms" for Computational Complexity and Computation of Finite Functions. *Information and Control*, 19, 377–386, 1971.

[30] Li Xiang. Effective immune sets, program index sets and effectively simple sets. In *Southeast Asian Conference on Logic*, New York, 1981.