CSC 202 Mathematics for Computer Science
Lecture Notes

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Chapter 9

Numbers, Cards, and Magic

9.1 Figurative Numbers

There is a famous story about the ten-year old Carl Friedrich Gauss and his teacher assigning the sum

$$1 + 2 + 3 + \ldots + 99 + 100$$

to class. While the rest of the students began adding up the numbers, Gauss immediately solved the problem, by rearranging the terms as follows:

$$\begin{aligned} & (1+100)+(2+99)+(3+98)+\ldots+(50+51) \\ = \; & \underbrace{101+101+\ldots+101}_{50} \\ = \; & 50 * 101 = 5050, \end{aligned}$$

destroying any chance the poor teacher had of a quiet repose. The basic idea behind Gauss's solution is worth noting: whenever you are dealing with something that varies (such as the numbers varying from 1 through 100) look for something that does not change. In this case, Gauss observed that the sum of the first and the last number is the same as the sum of the second and the second-to-last number, etc. and was able to use this to find the sum of the numbers quickly.

Let us try to generalize Gauss' observation. What is

$$1 + 2 + 3 + \ldots + (n-1) + n \quad ?$$

Let us assume for the moment that n is even, then using the same argument as above, we obtain:

$$\begin{aligned} & (1+n)+(2+(n-1))+(3+(n-2))+\ldots+(n/2+(n/2+1)) \\ = \; & \underbrace{(n+1)+(n+1)+\ldots+(n+1)}_{n/2} \\ = \; & n/2 * (n+1) = n(n+1)/2. \end{aligned}$$

If n is odd, we have to be a bit more careful: in that case there are $(n-1)/2$ pairs, each of them with a sum of $n+1$ and one extra term, $(n+1)/2$, the middle term:

$$\begin{aligned} & (1+n)+(2+(n-1))+(3+(n-2))+\ldots+((n-1)/2+(n+3)/2)+(n+1)/2 \\ = \; & \underbrace{(n+1)+(n+1)+\ldots+(n+1)}_{(n-1)/2}+(n+1)/2 \\ = \; & (n-1)/2*(n+1)+(n+1)/2=n(n+1)/2. \end{aligned}$$

The same result! We conclude that

$$1+2+3+\ldots+(n-1)+n=n(n+1)/2.$$

Euler introduced a simple way of writing sums with a variable number of terms, using $\sum$ to denote the sum. In this notation, we have shown that

$$\sum_{i=1}^{n} i=n(n+1)/2.$$

More generally, we write

$$\sum_{i=1}^{n} t_i$$

as a shorthand for

$$t_1+t_2+\cdots+t_n,$$

where t_i is a term depending on i such as i itself or i^2, $1/i$, 2^i and so on.

There is something unsatisfactory in our proof of this identity: why did we need to distinguish between the case that n is even and n is odd? The result turns out to be the same, an indication that we are not looking at the proof the right way. Indeed, there is a better way of looking at it suggested by a visual interpretation. Consider the following picture:

```
o
oo
ooo
oooo
...
ooooooo
```

How many pebbles are there? The answer is $\sum_{i=1}^{n} i$, the number we are interested in. How does visualizing the problem help? Based on the earlier idea of finding something constant in the rows of pebbles, let us add another set completing a rectangle:

```
o oooooooooo
oo ooooooooo
ooo oooooooo
...
oooooooooo o
```

The number of pebbles is (by inspection) $n(n+1)$, which means that $\sum_{i=1}^{n} i$, which is precisely half of the pebbles, must be $n(n+1)/2$. We can also phrase the proof more algebraically, but notice that it is the same proof:

$$\begin{aligned}
2\sum_{i=1}^{n} i &= \sum_{i=1}^{n} i + \sum_{i=1}^{n} i \\
&= \sum_{i=1}^{n} i + \sum_{i=1}^{n} (n+1-i) \\
&= \sum_{i=1}^{n} (i + (n+1-i)) \\
&= \sum_{i=1}^{n} (n+1) \\
&= n(n+1).
\end{aligned}$$

And, therefore, $\sum_{i=1}^{n} i = n(n+1)/2$.

Exercise 9.1.1. (i) What is the sum of the first n odd numbers? Give a geometric and an algebraic argument.

(ii) What is the sum of the first n even numbers? Give a geometric and an algebraic argument.

Counting the number of pebbles in geometric arrangements used to be an active research area in mathematics called *figurative numbers*. Indeed, $T(n) := n(n+1)/2$ is known as the nth triangular number, because it counts the number of pebbles in a triangle with n rows, here, for example, for $n = 4$:

```
   o
  o o
 o o o
o o o o
```

For obvious reasons, $S(n) := n^2$ is known as the nth square number.

Exercise 9.1.2. Show that $T(n)+T(n+1) = S(n+1)$ for all n; give a geometric and an algebraic argument.

The exercise shows that every square number is the sum of two triangular numbers. There are many other relationships like this between figurative numbers.

Exercise 9.1.3. Show the following equalities by giving two arguments: one geometric, one algebraic.

(*i*) $3T(n) + T(n-1) = T(2n)$.

(*ii*) $3T(n) + T(n+1) = T(2n+1)$.

(*iii*) $8T(n) + 1 = S(2n+1)$.

There is no reason to stop with squares, we can introduce pentagonal numbers, $P(n)$, and hexagonal numbers, $H(n)$, and so on; Figure 9.1 shows the first pentagonal numbers, $1, 5, 12, 22, \ldots$.

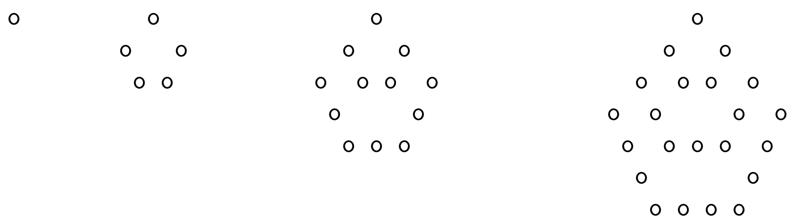


Figure 9.1: Pentagonal numbers: $P(1), P(2), P(3)$, and $P(4)$.

Exercise 9.1.4. Show that $P(n) = n + 3T(n-1)$ for all n; give a geometric argument. Conclude that $P(n) = n(3n-1)/2$.

The first hexagonal numbers, $H(n)$, are $1, 6, 15, 28, \ldots$, see Figure 9.2.

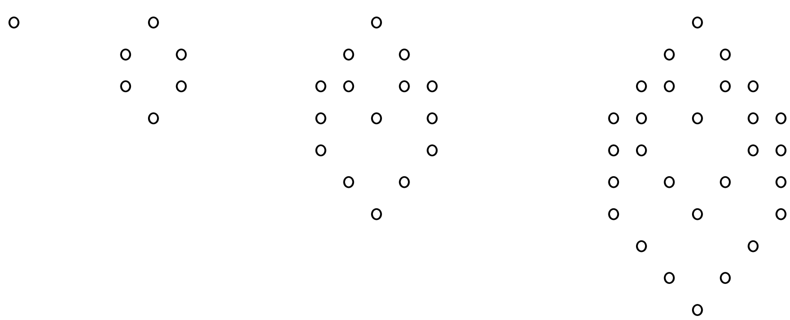


Figure 9.2: Hexagonal numbers: $H(1), H(2), H(3)$, and $H(4)$.

Exercise 9.1.5. Show that $H(n) = n + 4T(n-1)$ for all n; give a geometric argument. Conclude that $H(n) = n(2n-1)$.

If we consider our figurative numbers from a recursive point of view, they all have a simple definition:

$$T(n) = T(n-1) + n,$$

$$Q(n) = Q(n-1) + 2n - 1,$$
$$P(n) = P(n-1) + 3n - 2,$$
$$H(n) = H(n-1) + 4n - 3.$$

In other words, these numbers are the partial sums of the following series:

$$1 + 2 + 3 + 4 + 5 \ldots,$$
$$1 + 3 + 5 + 7 + 9 \ldots,$$
$$1 + 4 + 7 + 10 + 13 \ldots,$$
$$1 + 5 + 9 + 13 + 17 \ldots$$

An arithmetical series (or progression, or sequence) is a sequence $(a_i)_{1 \le i \le n}$ of numbers such that any two consecutive terms have the same difference. The sum $\sum_{i=1}^{n} a_i$ of an arithmetical series, where $a_1 = a$ and $a_i = a_1 + (i-1)d$ with d being the common difference between consecutive terms can be computed as

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (a + (i-1)d) = n(2a + (n-1)d)/2.$$

Exercise 9.1.6. Prove that the sum of the n term arithmetical series with first term a and difference d is

$$n(2a + (n-1)d)/2.$$

Similar methods work for higher powers; for example, let us determine

$$Py(n) = \sum_{i=1}^{n} i^2,$$

the nth square pyramidal number. (Visualize the first few terms by drawing square pyramids with n levels; the nth level is $S(n)$). Actually, before doing this, it is helpful to determine the tetrahedral number $Te(n)$, that is, the numbers of pebbles in a tetrahedron of side length n. $Te(n)$ can also be defined by $Te(n) = Te(n-1) + T(n)$ (a tetrahedron of height n consists of a tetrahedron of height $n-1$ together with one triangular layer of height n). Or, in other words, $Te(n) = \sum_{i=1}^{n} T(i)$. The first few tetrahedral numbers are $1, 4, 10, 20, \ldots$.

Let us look at $Te(4)$, for example; the layers of the tetrahedron are:

```
   1
  1 1          1
 1 1 1        1 1        1
1 1 1 1      1 1 1      1 1      1
```

Let us put those on top of each other:

```
   1
  1 2
 1 2 3
1 2 3 4
```

How can we add these numbers? We use our earlier trick by taking three of these and adding them the right way:

```
   1               1                4              6
  1 2             2 1              3 3            6 6
 1 2 3     +     3 2 1     +      2 2 2    =     6 6 6
1 2 3 4         4 3 2 1          1 1 1 1        6 6 6 6
```

That is, $3Te(4) = 6T(4) = 60$, so $Te(4) = 20$, which we already know to be true. This proof works in general, and it shows that

$$3Te(n) = (n+2)T(n),$$

in other words,

$$Te(n) = n(n+1)(n+2)/6.$$

We are now ready to compute

$$Py(n) = \sum_{i=1}^{n} i^2.$$

We know that $i^2 = S(i) = T(i) + T(i-1)$, so at each level a pyramid is the sum of two triangles; summing both sides from from 1 to n gives us that $Py(n) = Te(n) + Te(n-1)$, so

$$Py(n) = n(n+1)(2n+1)/6.$$

We conclude that

$$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6.$$

9.2 Fractals and Geometric Series

Let us look at a more geometric construction. We start with the line segment from $(0,0)$ to $(1,0)$ on the real line; we erect an equilateral triangle on the middle third $(1/3,0)$ to $(2/3,0)$ of the line. In the next step we erect a smaller equilateral triangles on each of the middle thirds of the f sides of the new triangle. And so on; in each step we a new triangle on the middle third of each side. The following picture illustrates the first few steps of the construction.

If we continue this process indefinitely, we obtain a geometric figure called a *fractal*.

```
0: ______________

         /\
        /  \
1: ____/    \____


       __/\__
       \    /
2: _/\_/    \_/\_
```

Exercise 9.2.1. Here is an exercise that shows that the figure we constructed is different from other geometric figures: what is the total length of the fractal line we are constructing? *Hint:* Calculate the total length for the nth step.

What's the area of the resulting figure? The whole fractal is safely contained in the square with corners $(0, 0)$ and $(1, 1)$, so the area is less than 1. The triangle we build in step 1 has area $A = 1/(24\sqrt{3})$. The triangles in step 2 have size $A/9$ (they are scaled copies of the original triangle, where the scaling factor in both x and y direction is $1/3$, so the area scales by a factor of $1/9$). Similarly, the triangles in the third step have size $A/9^2$. In general, the triangles we add in the nth step have size $A/9^{n-1}$. Moreover, there are 4^{n-1} triangles added in the nth step (including the first), since each side turns into four sides by the addition of a triangle. That is, in the nth step we add an area of $4^{n-1}A/9^{n-1} = A(4/9)^{n-1}$. Hence, the total area of the figure is

$$\sum_{n=1}^{\infty} A(4/9)^{n-1} = A \sum_{n=0}^{\infty} (4/9)^n,$$

using the distributive law (which holds for infinite sums as well). We can ignore the factor A for the moment and concentrate on the remaining infinite sum. It has the form

$$\sum_{n=0}^{\infty} x^n,$$

for some $0 < x < 1$ ($x = 4/9$ in our case). Let us call that sum S, that is,

$S = \sum_{n=0}^{\infty} x^n$. Now

$$\begin{aligned} xS &= x\sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^{n+1} \\ &= \sum_{n=1}^{\infty} x^n \\ &= S-1. \end{aligned}$$

So $xS = S - 1$, implying that $S = 1/(1-x)$.

$$\sum_{n=0}^{\infty} x^n = 1/(1-x) \text{ for } 0 < x < 1.$$

In our figure, we have $x = 4/9$, so $S = 9/5$, and the total area of the fractal is

$$9/5A = 9/(120\sqrt{3}) = \sqrt{3}/40,$$

which, roughly, is 0.04.

Remark 9.2.2. Here is a different way of looking at the argument: let us call the area of the fractal S. Now argue with self-similarity: if you look at the four sub-figures we build on the four sides of the very figure after the first step, their areas, because of self-similarity, must be $S/9$ each; so $S = A + 4S/9$, where A is the area of the triangle. Solving for S gives us the same solution as we got from solving the infinite sum.

The general *geometric series* has terms of the form $a_n = ax^n$, so $a_0 = a$ is the base term of the series which develops by multiplying a constant factor x in each step: $a_{n+1} = xa_n$. We can now easily compute the sum of the general geometric series as

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ax^n = a\sum_{n=0}^{\infty} x^n = a/(1-x).$$

Exercise 9.2.3. Show that

(i) $\sum_{n=1}^{\infty} x^n = x/(1-x)$,

(ii) $\sum_{n=k}^{\infty} x^n = x^k/(1-x)$ (for any k),

(iii) $\sum_{n=1}^{\infty} nx^n = x/(1-x)^2$ (*Hint:* use the difference trick),

(iv) $\sum_{n=1}^{\infty} n^2x^n = x(1+x)/(1-x)^3$.

You will often encounter finite geometric series such as $\sum_{n=k}^{m} x^n$; these can easily be dealt with as special cases of the infinite series; for example:

$$\begin{aligned} \sum_{n=k}^{m} x^n &= \sum_{n=k}^{\infty} x^n - \sum_{n=m+1}^{\infty} x^n \\ &= x^k/(1-x) - x^{m+1}/(1-x) \\ &= (x^k - x^{m+1})/(1-x). \end{aligned}$$

Exercise 9.2.4. For the general geometric series $a_n = ax^n$ find a closed form for $\sum_{n=k}^{m} a_n$.

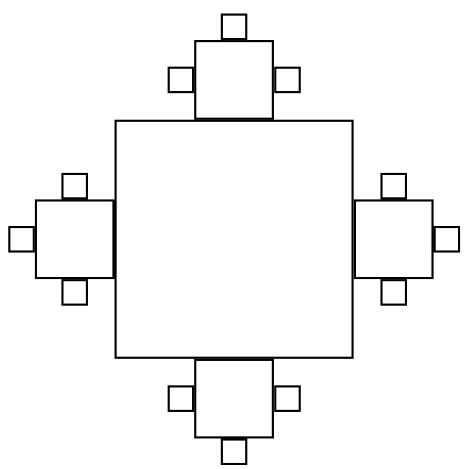
Remark 9.2.5. The applet at

`http://mathinsite.bmth.ac.uk/applet/aglr/aglr.html`

presents arithmetic and geometric series visually and let's you modify the parameters a, d and n to get a feel for how these series behave.

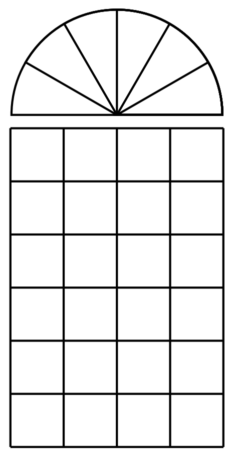
9.3 Exercises

1. We create a fractal as follows: start with a 1×1 square. On each of the four sides add a $1/3 \times 1/3$ square in the middle of the side. Now repeat: add a $1/9 \times 1/9$ square on each of the three free sides of the four new squares. The first three steps of constructing the fractal are shown below:



What's the area of the fractal? *Hint:* note that the number of free sides changes from the first to the second step.

2. While walking through Chicago you saw a nice ornamental window which looked like this:



How many squares do you see? *Hint*: The correct answer is not 24. Also, ignore the round part on the top, it is just for ornamentation.

3. Look again at the window from the previous exercise. How many rectangles do you see? If the window consisted of $n \times m$ basic squares, how many rectangles would you see?

4. How many edges does the complete graph K_n on n vertices have?

5. [The 7/8 puzzle] This is a 3×3 version of the 14/15 puzzle. You have a wooden frame holding square tiles numbered 1 through 8 and arranged as follows:

```
-------------
| 1 | 2 | 3 |
-------------
| 4 | 5 | 6 |
-------------
| 8 | 7 |   |
-------------
```

Can you move the tiles to obtain

```
-------------
| 1 | 2 | 3 |
-------------
| 4 | 5 | 6 |
-------------
| 7 | 8 |   |
-------------
```

or can you prove that this is not possible?

6. You are given a bar of chocolate consisting of n rows and m columns of chocolate squares. -picture You can break a bar at a row or a column. How many steps does it take to break the whole bar into squares of size 1? *Note:* You can only break one piece at a time.

7. The previous chapter contained the following puzzle by H.E. Dudeney:

   > The Dobsons secured apartments at Slocomb-on-Sea. There were six rooms on the same floor, all communicating, as shown in the diagram. The rooms they took were numbers 4, 5, and 6, all facing the sea. But a little difficulty arose. Mr. Dobson insisted that the piano and the bookcase should change rooms. This was wily, for the Dobsons were not musical, but they wanted to prevent any one else playing the instrument. Now, the rooms were very small and the pieces of furniture indicated were very big, so that no two of these articles could be got into any room at the same time. How was the exchange to be made with the least possible labour? Suppose, for example, you first move the wardrobe into No. 2; then you can move the bookcase to No. 5 and the piano to No. 6, and so on. It is a fascinating puzzle, but the landlady had reasons for not appreciating it. Try to solve her difficulty in the fewest possible removals with counters on a sheet of paper.

```
------------------------------
| Cabinet|         |  Piano  |
|                            |
|1       |2        |3        |
---    -----     -----     ---
|        |         |         |
|Drawers  Wardrobe   Bookcase|
| 4      |5        |6        |
------------------------------
```

   Show that there is no solution to this problem that finishes with the Cabinet in room 1, the Drawers in room 4 and the Wardrobe in room 5.

8. [Interview Question] You have b boxes and n dollars. Distribute the money in the boxes so that you can respond to any request for 0 to n dollars, by handing over some of the boxes without opening them to change their content. What are the restrictions on b and n and how do you distribute the money?

9. (The Baltimore Hilton Inn problem.) Your hotel room has an electronic lock with a keypad on which you can type digits 0 through 9. Your access

code is a four digit sequence. The door opens whenever the last four digits that have been typed in are the correct access code.

10. How many four-digit access codes are there?

11. You forgot your access code. Can you do better than type out all four-digit access codes? How many digits do you need to type in in the worst case?

12. There is a cheaper hotel around the corner in which the keypad only has digits 0, 1, and 2 and your access code consists of three digits. Write down the shortest sequence of digits that you can construct that will always open the door.

9.4 Notes and Additional Reading

The material in the section on figurative numbers is mostly drawn from John H. Conway and Richard K. Guy's impressive *The Book of Numbers*, which contain everything you ever wanted to know about numbers but were afraid to ask.