Strong Hanani-Tutte on the Projective Plane

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April 27, 2009

Abstract

If a graph can be drawn in the projective plane so that every two non-adjacent edges cross an even number of times, then the graph can be embedded in the projective plane.

1 Introduction

In the plane there is a beautiful characterization of planar graphs known as the Hanani-Tutte theorem: a graph is planar if and only if it can be drawn in the plane so that every two non-adjacent edges cross an even number of times. Equivalently, any drawing of a non-planar graph in the plane must contain two non-adjacent edges that cross oddly.

There are several proofs of the Hanani-Tutte theorem, including the original 1934 proof by Hanani and the 1970 proof by Tutte, see [7] for more references. Our goal in the current paper is to show that the result remains true in the projective plane.[1]

Theorem 1.1. *If a graph G can be drawn in the projective plane so that every two non-adjacent edges cross evenly, then G can be embedded in the projective plane.*

This is not the first result that indicates that the Hanani-Tutte theorem is not a special property of the plane. Using homology theory, Cairns and Nikolayevsky [2] showed that if a graph can be drawn on an orientable surface

[1] A sphere with a crosscap. We assume that the reader is familiar with the basic terminology of drawings and embeddings in surfaces. For background see [6, 3].

so that every pair of edges (not just non-adjacent ones) crosses an even number of times, then the graph can be embedded in that surface. Pelsmajer, Schaefer, and Štefankovič [8] gave a new, elementary proof of this weak Hanani-Tutte theorem that also establishes the result for non-orientable surfaces. Theorem 1.1 is the first time the strong version of the Hanani-Tutte theorem has been established for any higher-order surface.

There is an alternative view of the Hanani-Tutte theorem in terms of crossing numbers. The *crossing number* of a graph G, denoted by $\mathrm{cr}_S(G)$, is the minimum number of crossings in any drawing of G in surface S. Hence a graph G is embeddable in S if and only if $\mathrm{cr}_S(G) = 0$. The *odd crossing number* of G, denoted by $\mathrm{ocr}_S(G)$, is the minimum number of pairs of edges that cross oddly in any drawing of G in surface S. The *independent odd crossing number* of G, $\mathrm{iocr}_S(G)$, is the minimum number of pairs of non-adjacent edges that cross oddly in any drawing of G in surface S.

The strong Hanani-Tutte theorem can now simply be stated as "$\mathrm{iocr}(G) = 0$ implies $\mathrm{cr}(G) = 0$" and Theorem 1.1 becomes "$\mathrm{iocr}_{N_1}(G) = 0$ implies $\mathrm{cr}_{\mathrm{N}_1}(G) = 0$" using N_1 as a symbol for the projective plane. The weak Hanani-Tutte theorem in this notation reads "$\mathrm{ocr}_S(G) = 0$ implies $\mathrm{cr}_S(G) = 0$" and is true for all surfaces S as we mentioned above. The crossing number point of view emphasizes the algebraic nature of the Hanani-Tutte theorem as argued by van der Holst in [9].

Our proof of the strong Hanani-Tutte theorem for the projective plane uses techniques we developed for the Hanani-Tutte theorem and related results in the plane and higher-order surfaces [7, 8] and combines them with ideas from Mohar and Robertson on embeddings in the projective plane [5]; see Section 2. The proof will not naturally extend to any surface other than the projective plane, since it makes use of the list of minimal forbidden minors for the projective plane.

2 From Embeddings to Drawings

In this section we develop the necessary tools to deal with drawings in the projective plane. Some of these tools are extensions of well-known results for embeddings. All of them will play an important role in the proof of the strong Hanani-Tutte theorem for the projective plane.

2.1 Basic Observations and Redrawing Tools

Recall that a closed curve is *contractible* if it can be contracted to a point. In the projective plane a closed curve is contractible if and only if it passes through the crosscap an even number of times.[2]

[2] Any one-sided (or non-contractible) curve can serve as the crosscap. "Passing through the crosscap" means crossing that curve.

Lemma 2.1. *Suppose that a graph drawn in the projective plane contains two vertices connected by three internally disjoint paths. Of the three cycles formed by pairs of paths, exactly one or exactly three are contractible.*

Proof. Let P_1, P_2, P_3 be the three paths. Call a path even (odd) if it passes through the crosscap an even (odd) number of times. Then a cycle is contractible if and only if it is formed by two paths of the same parity. By the pigeonhole principle, exactly two or three of the paths have the same parity. The lemma follows. $\square$

Lemma 2.1 is based on [6, Proposition 4.3.1].

For convenience, we say that a particular drawing of a graph is *iocr*-0 if no pair of non-adjacent edges crosses an odd number of times.

Lemma 2.2. *If a graph G drawn on the projective plane contains two vertex-disjoint non-contractible cycles, then the drawing is not iocr-0.*

Proof. In the projective plane any two non-contractible curves cross an odd number of times.[3] Therefore there must exist two edges in G, one in each of the two cycles, that cross oddly. These must be non-adjacent, as they belong to vertex-disjoint cycles, so the given drawing of G is not iocr-0. $\square$

We will occasionally apply redrawing moves that lead to self-intersections of edges. These can be removed as shown in Figure 1.

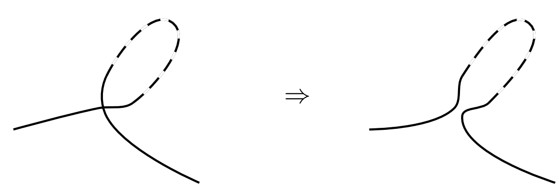

Figure 1: Removing a self-intersection; illustration from [7].

A ΔY*-exchange* in G is a process that replaces a triangle in a drawing of G with a claw (a $K_{1,3}$). The three vertices of the triangle become the leaves of the claw.

Lemma 2.3. *Let G be a graph with $\mathrm{iocr}_{N_1}(G) > 0$, and suppose G' can be obtained from G by a ΔY-exchange. Then $\mathrm{iocr}_{N_1}(G') > 0$.*

Proof. Consider an iocr-0 drawing of G'. Let e_1, e_2 and e_3 be the three edges of the claw. Draw a new edge f_1 by closely following e_1 and e_2; similarly add f_2 following e_2, e_3 and f_3 following e_3, e_1. Remove any self-intersections as shown in Figure 1. If f_1 crosses an edge e of $G' - \{e_1, e_2, e_3\}$ oddly, then e must cross

[3] Letting the first curve serve as the crosscap, we know that the other curve must pass through the crosscap an odd number of times since it is non-contractible.

either e_1 or e_2 oddly; hence e is incident to f_1. Similarly f_2 and f_3 only cross adjacent edges oddly. Removing e_1, e_2, e_3 now yields an iocr-0 drawing of G, which implies that $\mathrm{iocr}_{N_1}(G) = 0$. $\square$

The following lemma shows, roughly speaking, that for an iocr-0 drawing it is not a single vertex that makes the difference between planarity and non-planarity.

Lemma 2.4. *Let $x \in V(G)$ and $H = G - x$. Suppose there is an iocr-*0 *drawing of G in the projective plane such that all cycles in H are contractible. Then G is planar.*

Proof. Consider the specified drawing of G in the projective plane.

Claim: We can redraw G so that each edge of H passes through the crosscap an even number of times, and the drawing is still iocr-0.

Let F be a spanning forest of H and select a root in every component of F. Process the edges of F in a breadth-first order as follows: suppose uv is an edge of F so that u is closer to the root of uv's component than v. Start contracting uv by moving v along uv towards u, pushing all crossings along with it; stop when uv passes through the crosscap an even number of times. Call *uv processed*. Note that this move does not change the parity of how often any processed edge of F other than uv passes through the crosscap, since the only edges whose parity is changed by the contraction are edges incident to v, and none of those can have been processed already. At the end, every edge of F passes through the crosscap an even number of times. Every edge in $E(H) - E(F)$ also uses the crosscap evenly, since it completes a contractible cycle with some edges in F.

We now remove the crosscap and replace it with a disk. We reconnect severed edges by simple curves within the newly added disk.

Any two such curves within the disk have to cross oddly, since their crossings with the disk boundary alternate. Since each edge of H passes through the disk an even number of times, its crossing parity with every other edge does not change. Hence, any two edges whose crossing parity changes must be incident to x, which means that the independent odd crossing number is not affected by replacing the crosscap with a disk. We have thus obtained an iocr-0 drawing of G in the sphere (and, thereby, the plane), which implies that the graph is planar by the Hanani-Tutte theorem for the plane. $\square$

2.2 K-graphs and iocr

Let H be a subgraph of a graph G. An *H-component* or *H-bridge* is either an edge (and its endpoints) that does not belong to H but both of whose endpoints do, or a connected component of $G - V(H)$ together with all edges (and their endpoints) connecting this component to H.

H is a *K_4-graph* of G if it is a subdivision of K_4, and there is an H-component that is attached to all the vertices of degree 3 in H. H is a *$K_{2,3}$-graph* of G if it consists of three internally disjoint paths connecting two vertices, x and y, and there is an H-component that contains at least one internal vertex of each of the three paths. A K-*graph* is either a K_4-graph or a $K_{2,3}$-graph.

The following result is a well-known fact for embeddings (see [5, p326]). We relax the assumption that G is embedded to allow crossings, but control the crossings by requiring $\mathrm{iocr}_{N_1}(G) = 0$.

Lemma 2.5. *Let G be a graph containing a K-graph. Then every iocr-0 drawing of G in the projective plane contains two non-contractible cycles in the K-graph.*

Proof. Fix an iocr-0 drawing of G in the projective plane; let H be the K-graph within G. Note that if a K-graph contains one non-contractible cycle, then it must contain two by Lemma 2.1. Hence, for a contradiction, we may assume that all the cycles in H are contractible.

Suppose first that H is a subdivision of a $K_{2,3}$. Then H is the union of three internally disjoint paths P_1, P_2, P_3 that have the same endpoints x and y, and there is an H-component B containing an internal vertex of each path. Consider three edges from these vertices of H to $V(B) - V(H)$, and let T be a minimal tree in B that contains those edges. Then T has three leaves, so it must contain a unique vertex z of degree 3. Let $G' = H \cup T$. By construction, G' is a subdivision of $K_{3,3}$. The drawing of G yields an iocr-0 drawing of G'. All the cycles in $G' - z$ are cycles in H, which are contractible by assumption. Applying Lemma 2.4 implies that G' is planar, which is a contradiction.

If H is a subdivision of a K_4, let S be the set of its 4 vertices of degree 3 and let B be an H-component that contains S. As above, there is a tree T in B such that the set of its leaves is S. Since T has four leaves it either has a unique vertex of degree 4 or two vertices of degree 3. Let $G' = H \cup T$. Since G' contains a K_5-minor it is not planar. If T contains a vertex z of degree 4 we proceed as in the case of $K_{2,3}$: the cycles in $G' - z$ are cycles in H and therefore contractible, but then Lemma 2.4 implies that G' is planar, which is a contradiction. If T contains two vertices u, v of degree 3, let Q be the path between u and v in T. Let uw be the first edge in the path (possibly $w = v$). Contract uw by moving u along uw to w and then identifying u and w. As we contract, we may create odd crossings and self-intersections. Only drawings of edges incident to u may contain self-intersections; remove them as shown in Figure 1. Any new odd crossing will be between an edge e_1 that was incident to u before the contraction, and an edge e_2 that had crossed uw oddly. But e_2 must have been incident to either u or w, since the drawing was iocr-0. After contraction both edges are incident to $u = w$, so the drawing is still iocr-0. In this fashion we can contract all the edges along the path Q until we have a single vertex of degree 4, with the drawing of H unchanged. Then we are back in the first case, which suffices. ☐

Apart from applying Lemma 2.5 directly, we will also use it in the following variant.

Corollary 2.6. *If a graph G contains two disjoint K-graphs, then $\mathrm{iocr}_{N_1}(G) > 0$.*

Proof. Assume $\mathrm{iocr}_{N_1}(G) = 0$. Applying Lemma 2.5 twice gives us two vertex-disjoint non-contractible cycles, which contradicts Lemma 2.2. $\square$

3 Proof of the Main Theorem

If G cannot be embedded in the projective plane, then it must contain at least one of 35 minimal forbidden minors for the projective plane determined by Archdeacon, Glover, Huneke, and Wang [1, 4]. For a complete list of minimal forbidden minors and their names see [5][4] or [6, p198]. We show that all these graphs have independent odd crossing number larger than zero. This establishes Theorem 1.1 since from an iocr-0 drawing of a graph, one can easily obtain an iocr-0 drawing of any minor of the graph.[5]

The first twelve graphs are formed from two Kuratowski graphs by a disjoint union, a one-vertex identification, or a two-vertex identification and possibly deleting an edge between these vertices. It is easy to see that each contains two disjoint K-graphs. By Corollary 2.6 the independent odd crossing number of each of these twelve graphs is nonzero.

Of the remaining 23 minimal forbidden minors, C_7, E_{19}, D_{12}, E_{11}, E_{27}, D_9, G_1 [5, Fig. 3] and D_{17}, E_{20}, F_4 [5, Fig. 6] also contain two disjoint K-graphs, as observed in [5]. Again, by Corollary 2.6, the independent odd crossing number is nonzero for all of these graphs.

It is also known that each graph B_7, C_4, C_3, D_2 and E_2 [5, Fig. 6] can be obtained from graph A_2 through a sequence of ΔY-exchanges, and the graph E_5 can be obtained from the graph D_3 in the same way [5]. By Lemma 2.3 we need only show that $\mathrm{iocr}_{N_1}(D_3) > 0$ and $\mathrm{iocr}_{N_1}(A_2) > 0$ to prove a nonzero independent odd crossing number for all of these graphs.

Thus we are left with seven graphs, E_{22}, A_2, D_3, F_1, B_1, E_{18}, and E_3. For each we will assume an iocr-0 drawing in the projective plane, then find a contradiction.

Consider E_{22}, letting x be its unique degree 4 vertex as seen in Figure 2. Every 4-cycle not containing x is disjoint from a $K_{2,3}$-graph, so it must be contractible, by Lemma 2.2. For any two 4-cycles that share exactly one edge, their symmetric difference forms a 6-cycle, and by Lemma 2.1 this 6-cycle is contractible. Any other cycle in $E_{22} - x$ is the symmetric difference of one of those 6-cycles C and a 4-cycle C', such that $C \cap C'$ is a path; it is also contractible by Lemma 2.1. Then Lemma 2.4 gives a planar drawing of E_{22}, a contradiction.

We deal with A_2 by a similar argument. Let x be the unique degree 6 vertex in A_2 (see Figure 2). Any triangle not containing x is disjoint from a K_4-graph and is therefore contractible (Lemma 2.2). Suppose that C is a minimal non-contractible cycle in $A_2 - x$. If C is not an induced cycle, then Lemma 2.1 yields

[4] All references to [5] in this section are to the proof of Theorem 3.1 in that paper.

[5] We already used this in the proof of Lemma 2.5. This fact also underlies almost all proofs of the strong Hanani-Tutte theorem.

a shorter non-contractible cycle, contradicting minimality. If C is an induced cycle, then C is a 4-cycle and there is a vertex $y \neq x$ adjacent to every vertex of C. Let u and v be a pair of non-adjacent vertices in C. The union of either u, v-path in C and the path uyv forms a 4-cycle that is not induced, so it is contractible. Then C is contractible as well by Lemma 2.1, a contradiction. Hence, every cycle in $A_2 - x$ is contractible. Then Lemma 2.4 gives a planar drawing of A_2, a contradiction.

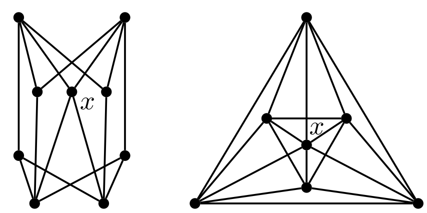

Figure 2: Vertex x in E_{22} (*left*) and in A_2 (*right*)

For graphs F_1 and D_3 we borrow part of an argument from [5]. The cycles $v_1v_2v_3v_4$ and $v_1v_2v_3u_1$ in F_1 (see Figure 3) are each disjoint from a $K_{2,3}$-graph, so they must both be contractible by Lemma 2.2. But the vertices v_1, v_2, v_3, v_4, u_1 induce a $K_{2,3}$-graph in F_1, and at most one of its three cycles is contractible, which contradicts Lemma 2.5.

For D_3 we apply the same argument to its cycles $v_1v_3v_2x$ and $v_1v_3v_2y$ (see Figure 3): Each is disjoint from a K_4-graph so both are contractible (Lemma 2.2). But there is a $K_{2,3}$-graph on vertices v_1, v_2, v_3, x, y, and at most one of its cycles is non-contractible, contradicting Lemma 2.5.

Next consider B_1. Any triangle containing exactly one of the vertices x, y, z is contractible since it is disjoint from a K_4-graph. Then by Lemma 2.1, the 4-cycles xu_1yu_2, xu_1zu_2, yu_1zu_2 are all contractible. But these are all the cycles in the $K_{2,3}$-graph on vertices $\{x, y, z\}, \{u_1, u_2\}$, which contradicts Lemma 2.5.

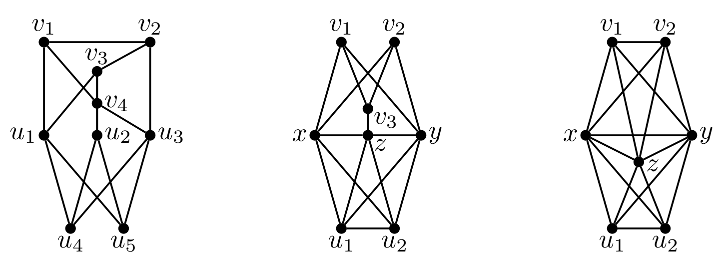

Figure 3: F_1, D_3 and B_1 (left to right) with labels

For the two remaining graphs we employ counting arguments. First consider E_{18}, which is $K_{4,4}$ with one edge removed (see Figure 4). Let $\{u_1, u_2, u_3, u_4\}$,

$\{v_1, v_2, v_3, v_4\}$ be the two partite sets and let u_4v_4 be the missing edge. Each of u_4 and v_4 is contained in 9 induced 4-cycles. Furthermore, for each 4-cycle containing u_4 there is a disjoint 4-cycle containing v_4 [5]. Hence, at least one of the two cycles must be contractible by Corollary 2.6. So one of u_4, v_4, say u_4, belongs to at most 4 non-contractible cycles. But $\{u_4, v_i, v_j, u_i, u_j\}$ induces a $K_{2,3}$-graph for each distinct pair i, j in $\{1, 2, 3\}$, and by Lemma 2.5 each one contains two non-contractible cycles. Since all these cycles are pairwise distinct, there are at least 6 non-contractible 4-cycles containing u_4, a contradiction.

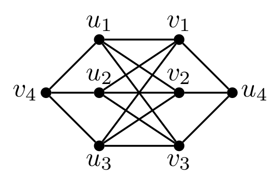

Figure 4: $E_{18} = K_{4,4} - u_4v_4$

Finally consider E_3, which is $K_{3,5}$. Let A, B be the two partite sets, with $|A| = 5$. By Lemma 2.5, for all $\{a_1, a_2, a_3\} \subset A$, $\{b_1, b_2\} \subset B$, exactly one of the three cycles in the $K_{2,3}$ induced by these 5 vertices is contractible. There are 30 possible choices of subsets of A and B as above, and every 4-cycle is present in 3 of the resulting subgraphs. Thus there are exactly 10 contractible 4-cycles in the given drawing of $K_{3,5}$. Some pair of vertices of B is shared by at least 4 of them, and two of these 4-cycles must also share a vertex of A. The union of these two 4-cycles is a $K_{2,3}$-graph, but by Lemma 2.5, the $K_{2,3}$-graph contains only one contractible cycle, a contradiction.

Acknowledgment. The authors wish to thank the anonymous referee, whose suggestion shortened the proof.

References

[1] Dan Archdeacon. A Kuratowski theorem for the projective plane. *Journal of Graph Theory*, 5(3):243–246, 1981.

[2] Grant Cairns and Yury Nikolayevsky. Bounds for generalized thrackles. *Discrete Comput. Geom.*, 23(2):191–206, 2000.

[3] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[4] Henry H. Glover, John Philip Huneke, and Chin San Wang. 103 graphs that are irreducible for the projective plane. *J. Comb. Theory, Ser. B*, 27(3):332–370, 1979.

[5] Bojan Mohar and Neil Robertson. Disjoint essential cycles. *J. Combin. Theory Ser. B*, 68(2):324–349, 1996.

[6] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[7] Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Removing even crossings. *J. Combin. Theory Ser. B*, 97(4):489–500, 2007.

[8] Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Removing even crossings on surfaces. In Jarik Nešetřil Alberto Márquez and Oriol Serra, editors, *European Conference on Combinatorics, Graph Theory and Applications (EuroComb '07)*, Electronic Notes in Discrete Mathematics, pages 85–90. Elsevier, August 2007.

[9] Hein van der Holst. Algebraic characterizations of outerplanar and planar graphs. *European J. Combin.*, 28(8):2156–2166, 2007.