Bounded Immunity and Btt-Reductions

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Abstract

We define and study a new notion called k-immunity that lies between immunity and hyperimmunity in strength. Our interest in k-immunity is justified by the result that $\emptyset'$ does not k-tt reduce to a k-immune set, which improves a previous result by Kobzev [7, 13]. We apply the result to show that $\emptyset'$ does not btt-reduce to MIN, the set of minimal programs. Other applications include the set of Kolmogorov random strings, and retraceable and regressive sets. We also give a new characterization of effectively simple sets and show that simple sets are not btt-cuppable.

Keywords: Computability, Recursion Theory, bounded reducibilities, minimal programs, immunity, k-immune, regressive, retraceable, effectively simple, cuppable.

1 Introduction

There seems to be a large gap between immunity and hyperimmunity (h-immunity) that is waiting to be filled. What happens, one wonders if the disjoint strong arrays that try to witness that a set is not h-immune are subjected to additional conditions, in particular a restraint on the cardinality of the sets in the array? First, let us recall the role immunity and thinness properties have played in computability theory[1].

In 1944 Post published his seminal paper on *Recursively enumerable sets of positive integers and their decision problems* (reprinted in Davis's *The Undecidable* [2]). With this paper he initiated what has since become known as Post's program: relating the thinness of a set to its degree. His goal was to show that there was a noncomputable c.e. set with a complement

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[1] In this paper we will use the terminology suggested by Soare [15]. In particular we will denote a computably (recursively) enumerable set by c.e. and talk about computable partial functions instead of partial recursive functions.

so thin that it could not be Turing complete. In 1958 Friedberg defeated Post's program for Turing completeness by constructing a Turing complete *maximal* set, i.e. a set with the thinnest possible complement. Notwithstanding this setback, Post's idea has been successful for stronger reductions. Post himself proved in his 1944 paper that a simple set cannot be btt-complete, and a hypersimple set not tt-complete. The first result was strengthened by Kobzev in 1973, when he proved that no set which is part of a computably inseparable pair of c.e. sets btt-reduces to a simple set. In turn our paper manages to improve on Kobzev by weakening the simplicity assumption.

We begin with the study of the mostly uncharted territory between immunity and h-immunity. A set A is called *k-immune* if there is no strong disjoint array for which every set in the array intersects A and has cardinality at most k. It is called *ω-immune* (or *bounded immune*) if it is k-immune for every k. We can then show that if B separates a computably inseparable pair of c.e. sets, then B does not k-tt-reduce to a k-immune set, and hence B does not btt-reduce to an ω-immune set. This gives a more detailed picture than the Kobzev result. Kobzev was interested in c.e. sets only, and c.e. sets are simple if and only if they have an ω-immune complement (this folklore result will be proved in Lemma 3.2). The concepts of k-immunity and bounded immunity have been used implicitly several times in the past, for example by Appel and McLaughlin [1] and Jockusch [6] in their work on retraceable and regressive sets.

We begin the paper with a short historical account of the success of Post's program. It contains no information essential to the rest of the paper, and the unhistorical reader can skip it without harm. In Section 3 we define k-immunity and establish some important basic results. The improved Kobzev result is presented in Section 4. We give several new applications, notably to MIN, the set of minimal programs (i.e. the minimal indices of a Gödel numbering), showing that $\emptyset'$ cannot btt-reduce to MIN. In Section 5 we study in some detail the consequences for retraceable and regressive sets. We go on to have a closer look at ω-immune sets which are not h-immune, and give a new characterization of effectively simple sets. Finally we show that simple sets are neither btt-cuppable nor d-cuppable.

For notation and definitions of the standard concepts of computability we refer the reader to the usual sources [13, 16]. In the following, degree always means Turing degree, unless specified otherwise.

2 A concise history of immunity and completeness

We collect the known results that connect notions of immunity with completeness in the following table. The first column contains the immunity assumption and the second column the conclusion. The last column contains references which can be checked in Odifreddi [13] and Soare [16]. The table presents the results as they are claimed in the reference, i.e. not necessarily the optimal result. We will make some remarks on that below the table. By B

separates a computably inseparable pair of c.e. sets we mean that there are disjoint c.e. sets A and C which are computably inseparable, and $A \subseteq B \subseteq \overline{C}$.

If A is	then	Proved in
co-immune,	$\emptyset' \not\leq_{\mathrm{m}} A$.	Post, 1944 [2]
immune, and B is c.e. and noncomputable,	$B \not\leq_{\mathrm{c}} A$.	easy exercise
co-immune, B separates a computably inseparable pair of c.e. sets,	$B \not\leq_{\mathrm{d}} A$.	Proposition 4.2
k-immune, and B separates a computably inseparable pair of c.e. sets,	$B \not\leq_{k\text{-tt}} A$.	Theorem 4.4
ω-immune, and B separates a computably inseparable pair of c.e. sets,	$B \not\leq_{\mathrm{btt}} A$.	Theorem 4.4
ω-immune, and B is c.e. and noncomputable,	$B \not\leq_{\mathrm{bd}} A$.	Schaefer, 1996 [15]
simple,	$\emptyset' \not\leq_{\mathrm{btt}} A$.	Post, 1944 [2]
simple, and B is part of a computably inseparable pair of c.e. sets,	$B \not\leq_{\mathrm{btt}} A$.	Kobzev, 1973 [7]
hypersimple,	$\emptyset' \not\leq_{\mathrm{tt}} A$.	Post, 1944 [2]
hyperimmune, and B is part of a computably inseparable pair of c.e. sets,	$B \not\leq_{\mathrm{tt}} A$.	Denisov, 1974 [3]
hypersimple,	$\emptyset' \not\leq_{\mathrm{wtt}} A$.	Friedberg, Rogers, 1959
hyperhypersimple,	$\emptyset' \not\leq_{\mathrm{Q}} A$.	Soloviev, 1974, and Gill, Morris, 1974
noncomputable, semirecursive, η-hyperhypersimple	$\emptyset' \not\leq_{\mathrm{T}} A$.	Degtev, 1973, and Marchenkov, 1976

The table prompts several natural questions some of which we will try to answer in the following notes.

Remarks.

(i) The Kobzev and the Denisov result as stated in the table both require B to be part of a computably inseparable set of c.e. sets. As a matter of fact both proofs (as presented by Odifreddi) will work with the weaker assumption that B separates a computably inseparable pair of c.e. sets. This includes non-c.e. sets as well.

(ii) A closer look at Odifreddi's proof of Denisov's result shows that for bounded truth-table reduction it establishes that if B separates a computably inseparable pair of c.e. sets and $B \leq_{k\text{-tt}} A$, then A is not $2k$-immune which is good enough to imply Kobzev's result, but not our Theorem 4.4.

(iii) Comparing the Denisov result with the Friedberg and Rogers result of the next line, one might wonder whether $\leq_{\mathrm{tt}}$ (in Denisov) can be improved to $\leq_{\mathrm{wtt}}$ while keeping the weaker assumptions of the Denisov result. This, however, is not possible, since every noncomputable c.e. wtt-degree contains a computably inseparable pair of c.e. sets [13, Proposition III.6.22]. On the other hand the Friedberg and Rogers result remains true with hyperimmunity instead of hypersimplicity; the usual proof shows that.

(*iv*) The Friedberg and Rogers result was improved by Downey and Jockusch [4]. By their result no hypersimple set is wtt-cuppable, namely if H is hypersimple there is no c.e. set B such that $H \oplus B$ is wtt-complete, unless B itself is already wtt-complete. A similar result has been proved by Nies and Sorbi for e-reductions [12]. We will show later that the btt-incompleteness of simple sets can be strengthened as well: simple sets are neither btt-cuppable, nor d-cuppable.

(*iv*) In 1976 Marchenkov (building on Degtev) finally came up with a property that forced Turing incompleteness. The property includes semirecursiveness which is not, strictly speaking, a thinness properties. Since then several other such properties—including $\mathcal{E}$-definable ones—implying Turing incompleteness have been found (Harrington and Soare [5]).

3 The notion of k-immunity

Let $(D_x)_{x\in\omega}$ be a canonical numbering of the finite sets. A family $(D_{f(x)})_{x\in\omega}$ of finite sets is a *disjoint strong array* if f is a computable function, and all the $D_{f(x)}$ are pairwise disjoint and nonempty. It is a *disjoint strong k-array* if it is a disjoint strong array, and in addition $|D_{f(x)}| = k$ for all $x \in \omega$. If we want to emphasize f, we say that f *describes a disjoint strong array (or k-array)*. For brevity we will also sometimes say that the *array intersects a set A*, which just means that every set in the array intersects A.

We can now define a notion of immunity between immune and h-immune.

Definition 3.1 *A set M is k-*immune *if there is no disjoint strong k-array, all of whose elements intersect M. That is, there is no computable f such that*

- $(\forall x)\ |D_{f(x)}| = k$,
- $(\forall x \neq y)\ D_{f(x)} \cap D_{f(y)} = \emptyset$, *and*
- $(\forall x)\ D_{f(x)} \cap M \neq \emptyset$.

*A set M is ω-*immune *(or* bounded immune*) if it is k-immune for all k.*

Note that $(k+1)$-immunity implies k-immunity for all $k \geq 1$, and 1-immunity is the same as immunity. This finer hierarchy of immunity has not attracted a lot of attention. One partial reason is without doubt the following folklore result.

Lemma 3.2 *If $\overline{A}$ is simple, then A is ω-immune.*

Proof. Suppose A is an immune set which is not k-immune for some $k > 1$, and $\overline{A}$ is c.e. We will prove that A is not $(k-1)$-immune, which proves the lemma. Let $(D_{f(x)})_{x\in\omega}$ be as in the definition of k-immune. Since A is immune there must be infinitely many x for which $D_{f(x)}$ intersects $\overline{A}$. Using that $\overline{A}$ is c.e., we can enumerate these infinitely many $D_{f(x)}$ dropping the element which was found to be in $\overline{A}$. So we can enumerate infinitely many finite sets that are pairwise disjoint, intersect A, and contain at most $k-1$ elements. In short, A is not $(k-1)$-immune. □

The notions of k-immunity can be separated in Δ^0_2 though. A much stronger result will be proved in the following section, and some natural examples are presented in Section 3.2.

3.1 Separating the new notions of immunity

In this section we will prove two theorems which show that k-immune sets that are not $(k+1)$-immune, and ω-immune sets that are not h-immune are—in a sense—abundant.

Theorem 3.3 *Every noncomputable c.e. degree contains a k-immune set which is not $(k+1)$-immune (for arbitrary k).*

This theorem is a consequence of a lemma which is stronger, but does not have the dashing good looks of the theorem.

Lemma 3.4 *For every noncomputable c.e. set B, and every k there is a k-immune set $A \leq_{\mathrm{wtt}} B$, such that B bounded-disjunctively reduces to A with norm $k+1$, and A is not $(k+1)$-immune.*

Proof. Fix k. We will construct the set A by a finite injury argument. Define a family of intervals $I_e := \{z : e(k+1) \leq z < (e+1)(k+1)\}$. The set A will contain at most one point in every I_e, and A will meet the requirement

$$P_e \quad : \quad e \in B \text{ iff } A \cap I_e \neq \emptyset,$$

for all e. This will ensure that B bounded-disjunctively reduces to A with norm $k+1$. Furthermore if C is an infinite computable subset of B, then the family $(I_e)_{e\in C}$ is a disjoint strong $(k+1)$-array which witnesses that A is not $(k+1)$-immune. (See also the remark after the proof.)

The set A will furthermore have to meet the requirements

$$\begin{aligned} N_e \quad : \quad & \text{if } (D_{\varphi_e(n)})_{n\in\omega} \text{ is a disjoint strong } k\text{-array,} \\ & \text{then there is an } n \text{ such that } D_{\varphi_e(n)} \text{ does not intersect } A, \end{aligned}$$

which make A k-immune.

The construction is a finite injury argument with regard to the N type of requirements. Requirement N_e has higher priority than $N_{e'}$ if $e < e'$. Requirements of type P will be fulfilled instantaneously.

During the construction the intervals will be assigned to requirements of type N. At each stage, every interval is assigned to at most one requirement. A requirement N_e has associated with it a *taboo set* T_e of elements it tries to keep out of A. A taboo set contains at most k elements.

Let f be a computable function enumerating B without repetitions.

Stage $s = 0$. Let $A^0 = \emptyset$. Initially all intervals are unassigned, and all the taboo sets T_e are empty.

Stage $2s + 1$. (Satisfy $P_{f(s)}$.) Let $e = f(s)$. We will satisfy P_e. The set $A^{2s} \cap I_e$ is currently empty, since f enumerates B without repetition, and only P_e can put an element into I_e. If I_e is currently assigned to a requirement $N_{e'}$, then let a be the smallest element in $I_e - T_{e'}$ (note that this is possible, since I_e contains $k + 1$ elements, whereas $T_{e'}$ contains only k). Should I_e not be assigned to a requirement let a be the smallest element of I_e. Put a into A^{2s+1}. Requirement P_e is satisfied, and will remain so henceforth.

Stage $2s + 2$. (Make A k-immune, and $A \leq_{\mathrm{wtt}} B$.) Say N_e *requires attention at stage* $2s + 2$ if there are no intervals assigned to N_e, and there exist t and x such that

(*i*) $\langle e, t, x\rangle < s + 1$, and

(*ii*) $|D_{\varphi_{e,t}(x)}| = k$, and

(*iii*) $D_{\varphi_{e,t}(x)}$ does not intersect any interval $I_{e'}$ with $e' < e$, nor does it intersect any interval assigned to a requirement $N_{e'}$ of higher priority, and

(*iv*) $f(s + 1) < \min(D_{\varphi_{e,t}(x)})$.

Choose the smallest e such that N_e requires attention at stage $2s + 2$. Say requirement N_e *receives attention at stage* $2s + 2$. Let $T_e = D_{\varphi_{e,t}(x)}$. Remove all assignments of intervals to requirements of lower priority than N_e, and assign all intervals that intersect T_e to N_e. For all intervals I_n that are assigned to N_e, and for which $I_n \cap A^{2s+1} \neq \emptyset$, let $I_n \cap A^{2s+2}$ contain only the smallest element of I_n that is not in T_e (again this is possible because of cardinality reasons). For all other intervals I_n let $A^{2s+2} \cap I_n = A^{2s+1} \cap I_n$. Let all $T_{e'} = \emptyset$ for $e' > e$. *End of Construction.*

If we assume that A^s converges to a set A, then we see that by construction all P_e are satisfied: if $e \notin B$, then there will never be an element in $A \cap I_e$; if $e \in B$, then an element is put into the corresponding interval (which is possible, because at any one time there is at most one taboo set associated with the interval, so there is an element of the interval not in the taboo set). During stages $2s+2$ this element might be moved within the interval, but this does not affect P_e (assuming that A exists).

Every requirement of type N can only be injured by requirements of type N of higher priority, and hence acts only finitely often. Since every interval can only be changed by finitely many requirements this implies that every interval will have a final assignment, or remain without an assignment. In either case $\lim_{s\to\infty} A^s \cap I_n$ exists for every n, and hence the A^s converge to a set A for $s \to \infty$.

Suppose that not all requirements of type N are satisfied. Choose N_e to be the requirement of highest priority that is not satisfied. Then by the argument of the preceding paragraph there is a stage s' after which no requirement $N_{e'}$ of higher priority will act, so N_e will not be injured after stage s'. There are no intervals assigned to N_e at stage s' and $(D_{\varphi_e(x)})_{x\in\omega}$ is a disjoint strong k-array (otherwise N_e would be satisfied, and remain satisfied).

Given y we can effectively find x and t such that conditions (ii) and (iii) of the construction are satisfied, and moreover $\min(D_{\varphi_e(x)}) > y$. Then at stage $s = \max\{s', \langle e, x, t\rangle\}$ all conditions except (iv) are satisfied. Since N_e cannot receive attention (otherwise it would be satisfied permanently), we can conclude that $y < \min(D_{\varphi_e(x)}) \leq f(s''+1)$ for all $s'' > s$. Hence $y \in B$ iff $y \in \{f(0), \ldots, f(s)\}$ which is impossible, since B is not computable. Therefore N_e is satisfied.

We are left with the proof that $A \leq_{\mathrm{wtt}} B$. But this follows from the permitting in condition (iv) together with (iii): given y determine the first interval I_n to the right of y (i.e. $y \notin I_n$ and $y \in I_{n-1}$), and use B to determine a stage s such that $\{f(0), \ldots, f(s)\} = B \cap \bigcap_{i<n} I_i$. Since f enumerates B without repetitions, from this stage on conditions (iii) and (iv) ensure that A does not change on $\bigcap_{i<n} I_i$, so we can decide A by simulating the construction up to stage s. This gives us a wtt-reduction from A to B. $\square$

Remark. The above proof showed explicitly that the constructed set A was not $(k+1)$-immune. This could have been avoided by using an observation made by Schaefer [14]: if a c.e. noncomputable set bounded disjunctively reduces to a set B with norm $(k+1)$, then the set B is not $(k+1)$-immune.

Using a very similar proof, bounded immunity can be separated from h-immunity. We just state the result without proof. The theorem and the lemma are immediate consequences of Lemma 6.4.

Theorem 3.5 *Every noncomputable c.e. degree contains an ω-immune set which is not h-immune.*

Again there is a stronger lemma from which the theorem follows.

Lemma 3.6 *For every noncomputable c.e. set B, there is an ω-immune set $A \leq_{\mathrm{wtt}} B$, such that B disjunctively reduces to A, and A is not h-immune.*

3.2 Kolmogorov random strings and minimal indices

After showing that our newly defined variant of immunity is non-trivial, it is time we presented some natural examples. The set R of Kolmogorov random strings is defined by $\mathrm{R} = \{x : (\forall y \leq x)[\varphi_y(0) \neq x]\}$.

Theorem 3.7 R *is ω-immune but not h-immune.*

We omit the straightforward proof. One might object that R is co-c.e. and bounded immunity collapses to immunity, so in Section 5 on retraceable and regressive sets we present an easy example of a set in Δ^0_2 which is not co-c.e., ω-immune and not h-immune.

Another natural example, which can be found higher up in the hierarchy, is MIN, the set of minimal indices of a Gödel numbering, i.e. $\mathrm{MIN} = \{e : (\forall i < e)\ [\varphi_i \neq \varphi_e]\}$. It is well known [11, 14] that MIN is Turing-equivalent to $\emptyset''$, and that it is not h-immune.

Theorem 3.8 (Schaefer [14]) MIN *is ω-immune.*

Proof. The proof generalizes the usual immunity proof for MIN using the k-fold Recursion Theorem. Suppose MIN was not k-immune. Let $D_{f(i)}$ witness this as in the definition of k-immunity. Define a computable function $h(x_1, \ldots, x_k) := f((\mu i)[(\forall z \in D_{f(i)})(\forall j)[z > x_j]])$. The function h picks out the index of the first set in the enumeration for which all elements are bigger than any x_j. We use h to define k computable functions. For $1 \leq i \leq k$ let

$$g_i(x_1, \ldots, x_k) := \text{the } i\text{th element of } D_{h(x_1,\ldots,x_k)} \ .$$

By the k-fold Recursion Theorem there are k indices $e_1, \ldots, e_k$ such that $\varphi_{g_i(e_1,\ldots,e_k)} = \varphi_{e_i}$ for all $1 \leq i \leq k$. Since $g_i(e_1, \ldots, e_k) > e_i$ this contradicts the fact that $g_i(e_1, \ldots, e_k)$ has to be a minimal index for some i. □

A careful look at the proof shows that MIN is in fact effectively k-immune: there is a total computable function g such that if W_e is a set of canonical indices of pairwise disjoint sets, all of which intersect MIN and contain at most k elements, then $g(e)$ is an upper bound on the cardinality of W_e (in fact it even bounds $\max_{x \in W_e}\{\min(D_x)\}$).

4 Bounded truth-table reducibility

Two sets A and B are *computably inseparable* if there is no computable set C for which $A \subseteq C \subseteq \overline{B}$.

Definition 4.1 *We call a set E a* separator *if it separates a computably inseparable pair of c.e. sets, i.e. there are c.e. sets A and B which are computably inseparable, and $A \subseteq E \subseteq \overline{B}$.*

It is obvious that a separator cannot be computable. As a consequence of the Low Basis Theorem by Jockusch and Soare [16] we know that there are low separators. Before we turn to bounded truth-table reducibilities we will first show an easy result on disjunctive reducibility that illustrates the power of computably inseparable sets.

Proposition 4.2 *If M is co-immune, and E is a separator, then $E \not\leq_{\mathrm{d}} M$.*

Proof. Let A and B be the pair of noncomputable c.e. sets separated by M, i.e. $A \subseteq M \subseteq \overline{B}$, and f computable such that $x \in E$ iff $D_{f(x)} \cap M \neq \emptyset$. Then $D_{f(x)} \subset \overline{M}$ for all $x \in B$, and since $\overline{M}$ is immune, $\bigcup_{x \in B} D_{f(x)}$ is finite. But this contradicts the inseparability of A and B. □

Since $\emptyset'$ is half of a computably inseparable pair, the following corollary is immediate.

Corollary 4.3 *If M is simple, then $\emptyset' \not\leq_{\mathrm{d}} M$.*

Contrast this with the fact that Post's construction of a simple set can be modified so as to yield a conjunctively complete simple set.

We now turn to our real concern in this section, the bounded reducibilities.

Theorem 4.4 *If E is a separator, and M is a k-immune set, then $E \not\leq_{k\text{-tt}} M$.*

This result was proved without knowledge of an earlier, weaker result by Kobzev [13, Exercise III.8.10], which uses similar techniques.

As above we get the following corollary.

Corollary 4.5 *For all $k \geq 1$ and any set M, if M is k-immune, then $\emptyset' \not\leq_{k\text{-tt}} M$.*

Using the Low Basis Theorem we can get another corollary to the theorem: there is a low set which does not k-tt reduce to any k-immune set. We will state all the corollaries for $\emptyset'$, but the results also hold for some low set. Note that the corollary is optimal with regard to k, since by Lemma 3.4 there is a k-immune set which is $(k+1)$-disjunctively equivalent to $\emptyset'$.

In particular, for the sets R and MIN as defined in the last section we have the following corollaries.

Corollary 4.6 (Kummer [9]) $\emptyset' \not\leq_{\mathrm{btt}} \mathrm{R}$.

Concerning weaker reductions Kummer proved the surprising fact that R is truth-table complete.

Corollary 4.7 $\emptyset' \not\leq_{\mathrm{btt}} \mathrm{MIN}$.

This result is optimal in the sense that there is a Gödel numbering relative to which MIN is truth-table complete for the second level of the arithmetical hierarchy [14]. It is an open question, however, whether MIN is truth-table complete for arbitrary Gödel numberings.

The proof of Theorem 4.4 breaks down into two parts. The general case of k-tt reductions is reduced to a fixed truth table by Lemma 4.9, and the fixed truth table case is resolved in Lemma 4.10. For the sake of clarity we include the definition of a reduction via a fixed truth table.

Definition 4.8 *Let E and M be arbitrary sets, let $\alpha\colon 2^k \to 2$ be a k-ary Boolean function, and let f be computable. We say that E α-reduces to M via f $(f\colon E \leq_\alpha M)$ if*

- $(\forall x)\ |D_{f(x)}| = k$,
- $(\forall x \in E)\ \alpha(\chi_M(D_{f(x)})) = 1$, *and*
- $(\forall x \notin E)\ \alpha(\chi_M(D_{f(x)})) = 0$,

where $\chi_M(D_y)$ is the vector $(M(y_1), \dots M(y_k))$, where $D_y = \{y_1 < \cdots < y_k\}$. (We say that $E \leq_\alpha M$ if there exists such an f.)

Lemma 4.9 *Suppose E is a separator and M an arbitrary set. If $E \leq_{k\text{-tt}} M$, then there is a separator $\widetilde{E}$ and a k-ary Boolean function α such that $\widetilde{E} \leq_\alpha M$.*

Proof. Let A and B be two computably inseparable c.e. sets with $A \subseteq E \subseteq \overline{B}$. Since $E \leq_{k\text{-tt}} M$, there is a computable f, and a k-ary Boolean function α_x effectively computable from x such that $|D_{f(x)}| = k$ for all $x \in \omega$, and

- $(\forall x \in A)\ \alpha_x(\chi_M(D_{f(x)})) = 1$, and
- $(\forall x \in B)\ \alpha_x(\chi_M(D_{f(x)})) = 0$.

Let $\ell = 2^{2^k}$ and let $\tau_1, \ldots, \tau_\ell$ enumerate all k-ary Boolean functions. For $1 \leq i \leq \ell$, set $T_i = \{x : \alpha_x = \tau_i\}$. If $A \cap T_i$ and $B \cap T_i$ are computably separable for all i, then clearly A and B are computably separable, so fix i such that $\widetilde{A} = A \cap T_i$ and $\widetilde{B} = B \cap T_i$ are computably inseparable, and set $\alpha = \tau_i$, and $\widetilde{E} = \{x : \alpha(\chi_M(D_{f(x)}) = 1\}$. $\square$

Lemma 4.10 *If E is a separator and M is k-immune, then there is no k-ary α such that $E \leq_\alpha M$.*

Proof. Let A and B be a computably inseparable pair of c.e. sets with $A \subseteq E \subseteq \overline{B}$. Suppose M is k-immune and there is a k-ary α such that $E \leq_\alpha M$ via some computable f. We can assume without loss of generality that $\alpha(\vec{0}) = 0$, since otherwise we consider $\neg\alpha$ and $\overline{E}$.

We proceed by *reductio ad absurdum* on k. If $k = 1$, then we must have $\alpha(1) = 1$ by the inseparability of A and B. Thus for all $x \in A$ we have $D_{f(x)} \subseteq M$, and for all $x \in B$ we have $D_{f(x)} \subseteq \overline{M}$. Since each $D_{f(x)}$ is a singleton and M is (1-)immune, $\bigcup_{x \in A} D_{f(x)}$ is finite, which contradicts the inseparability of A and B.

Now suppose $k > 1$, A and B are computably inseparable c.e. sets, M is k-immune, and $f\colon E \leq_\alpha M$. To complete the proof, we show that there is a separator $\widetilde{E}$ such that $\widetilde{E} \leq_\beta M$ for some $(k-1)$-ary β.

Again, assume without loss of generality that $\alpha(0^k) = 0$, so in particular $D_{f(x)} \cap M \neq \emptyset$ for all $x \in A$. There exists a finite set $S \subseteq \omega$ such that $\min(D_{f(x)}) \in S$ for all $x \in A$, since otherwise we could build a disjoint strong k-array of sets intersecting M, contradicting the k-immunity of M. Let $S_0 = S \cap \overline{M}$ and let $S_1 = S \cap M$. For $i \in 2$, define

$$A_i = \{x \in A : \min(D_{f(x)}) \in S_i\},$$

and

$$B_i = \{x \in B : \min(D_{f(x)}) \in S_i\}.$$

Note that all the A_i and B_i are c.e., $A_i \cap B_i = \emptyset$, and the A_i partition A.

Claim 4.11 *There is an $i \in 2$ such that A_i and B_i are computably inseparable.*

Proof of Claim. Suppose not. For $i \in 2$, let C_i be computable with $A_i \subseteq C_i \subseteq \overline{B_i}$. Let

$$C = \bigcup_{i \in 2} (C_i \cap \{x : \min(D_{f(x)}) \in S_i\}).$$

Clearly, C is computable and by definition $A \subseteq C \subseteq \overline{B}$, contradicting the inseparability of A and B. $\square$ **Claim**

Now fixing $i \in 2$ such that A_i and B_i are computably inseparable, we set $F = A_i$, $G = B_i$, and $\beta = \lambda\vec{v} \in 2^{k-1}.\alpha(i, \vec{v})$. Define g such that $D_{g(x)} = D_{f(x)} - \{\min(D_{f(x)})\}$, and finally let $\widetilde{E} := \{x : \beta(\chi_M(D_{g(x)})) = 1\}$. Then $F \subseteq \widetilde{E} \subseteq \overline{G}$, and $\widetilde{E}$ is a separator by the claim; furthermore $\widetilde{E} \leq_\beta M$ via g. □

Remark. It was observed by Fischer [13] that any btt-reduction can be transformed into one where the truth table is fixed, i.e., independent of the input. So if one is only interested in btt-reducibility, then we can avoid using Lemma 4.9 and prove that if M is k-immune for all k, then $\emptyset' \not\leq_{\mathrm{btt}} M$—a result somewhat weaker than Corollary 4.5, but still strong enough to prove Corollary 4.7. Lemma 4.10 (with Lemma 4.9 or Fischer's observation) also allows an easy proof of Post's result that no simple set can be btt-complete.

5 Retraceable and Regressive Sets

Definition 5.1 *A (partial) function g* regresses *the set A if there is a (not necessarily effective) enumeration $(a_n)_{n\in\omega}$ of A without repetitions such that $g(a_{n+1}) = a_n$ and $g(a_0) = a_0$. If the enumeration can be taken to be in increasing order, then g is said to* retrace *A. A set is called* regressive *(*retraceable*) if it is regressed (retraced) by a computable partial function.*

By the classic result of Dekker and Myhill a retraceable set is either computable or immune, and a regressive set is either c.e. or immune. Regarding regressive sets the following strengthening was shown by Appel and McLaughlin (stated in a different terminology naturally).

Theorem 5.2 (Appel and McLaughlin [1]) *A regressive set is either c.e. or ω-immune.*

Another theorem is immediate from this (although Appel and McLaughlin do not mention this).

Theorem 5.3 *A retraceable set is either computable or ω-immune.*

Proof. A retraceable set is regressive, so it is either c.e. or ω-immune. If it is c.e. it is not immune, and hence computable by the result of Dekker and Myhill. □

Corollary 5.4 *If A is retraceable, then $\emptyset' \not\leq_{\mathrm{btt}} A$.*

The theorem allows us to fulfill the promise of an easy example of a set in Δ^0_2 that is ω-immune without being co-c.e.: let A be the set of initial segments of the characteristic function of $\emptyset'$, i.e. $A = \{\sigma \in \{0,1\}^* : (\forall i < \mathrm{lh}(\sigma))[\sigma(i) = \emptyset'(i)]\}$, where $\mathrm{lh}(\sigma)$ is the length of the string σ. Then A is obviously retraced by a total computable function (which deletes the last bit). It is d.c.e., but neither c.e., nor co-c.e., and hence by the preceding theorem ω-immune. It is not h-immune, since it contains a string of every length.

This example also shows that every set has a truth-table equivalent retraceable set, which is optimal by the corollary.

Using a theorem of Kobzev and Lachlan [13] that every btt-complete set is bd-complete we can conclude with the help of Theorem 5.2:

Theorem 5.5 *If A is regressive and $\emptyset' \leq_{\mathrm{btt}} A$, then A is c.e., and hence bd-complete.*

This result is optimal, since $\emptyset'$ itself is c.e. and hence regressive. Bounded disjunctive completeness is also the best we can expect, as illustrated by the following example: $\emptyset'$ can be split into two disjoint low c.e. sets A and B (according to the Sacks Splitting Theorem). Then $\emptyset'$ disjunctively reduces to $A \times B$ with two queries. Furthermore $A \times B$ is a c.e. and hence regressive set, but it cannot be m-complete (not even 1-tt-complete), since otherwise one of A or B would have to be m-complete (by a result of Lachlan [16, Exercise II.4.16]).

Another notion which simultaneously generalizes enumerability and regressiveness was introduced by Jockusch [6]. A set A is called *uniformly introenumerable* if there is one e such that $\chi_A = \varphi_e^B$ for all infinite subsets B of A. Jockusch proved that every immune uniformly introenumerable set is ω-immune.

Theorem 5.6 *If A is immune and uniformly introenumerable, then $\emptyset' \not\leq_{\mathrm{btt}} A$.*

The classical results on regressive and retraceable sets roughly speaking show that these sets cannot have easy infinite subsets without being easy themselves. We end this section by a result which argues that the opposite view is also true. Remember that a set A is *effectively immune* if there is a (total) computable function f such that $W_e \subset A \Longrightarrow |W_e| < f(e)$ for all e.

Theorem 5.7 *Suppose A has an infinite subset computable in B. If A is regressive and effectively immune, then $A \leq_{\mathrm{T}} B$.*

Proof. Let C be the infinite subset of A with $C \leq_{\mathrm{T}} B$. Suppose A is regressed by the computable partial function g, and strongly effectively immune via f. Let $W_{h(e)} = \{e, g(e), g^2(e), \ldots\}$. Define $l(e) = f(h(e))$. To decide whether $e \in A$ search for an element $x \in C$ such that $g^{l(e)}(x) \neq g^{l(e)+1}(x)$ (such x's are abundant). Then $e \in A$ if and only if there is a k such that $g^k(x) = e$. Namely if $g^k(x) = e$ for some k, then $e \in A$ since $x \in A$ and g regresses A. For the other direction note that if $e \in A$ every regression sequence containing at least $l(e)$ elements which is started on an element of A must run through e. □

A set A is called *introreducible* if it is computable in all its infinite subsets, and *uniformly introreducible* if there is one oracle algorithm that computes A with any infinite subset of A as an oracle. Note that the above proof was uniform in C.

Corollary 5.8 *If a set is regressive and effectively immune, then it is uniformly introreducible.*

In Theorem 5.7 the condition that A be strongly effectively immune could have been dropped, to still get the conclusion that $A \leq_{\mathrm{T}} B \oplus \emptyset'$, since in the proof we can use the $\emptyset'$ oracle to compute an upper bound on $|W_{h(e)}|$.

Corollary 5.9 *Suppose A has an infinite subset computable in B. If A is regressive, then $A \leq_{\mathrm{T}} B \oplus \emptyset'$.*

This can be put more succinctly by letting A be Σ_k complete and $B = \emptyset^{(k-1)}$ for $k \geq 2$.

Corollary 5.10 *No Σ_k-complete set is regressive for $k \geq 2$.*

We can apply the corollary to MIN, since it is Σ_2-complete.

Corollary 5.11 MIN *is not regressive.*

6 Fat and thin sets

We already mentioned earlier (without proving it) that there are ω-immune sets which are not h-immune. There is still a very large gap between hyperimmunity and bounded immunity which we will investigate in this section. To do so we first introduce a new notation. Let $(D_{f(x)})_{x\in\omega}$ be a disjoint strong array. With this define $(\#f)(n) = |\{x : |D_{f(x)}| = n\}|$, i.e. $\#f$ counts how many sets of cardinality n appear in the array. Since we will restrict our attention to disjoint strong arrays intersecting ω-immune sets, $\#f$ only takes finite values. Remember that g *dominates* f if $g(n) > f(n)$ for almost all n. If the inequality holds only infinitely often, then g is said to *majorize* f.

Definition 6.1 *An ω-immune set A is called* thin *if there is a computable function which dominates every $\#f$, where $(D_{f(x)})_{x\in\omega}$ is a disjoint strong array intersecting A that is $A \cap D_{f(x)} \neq \emptyset$ for all x.*

The intuition is that although A is not h-immune, it is so thin that we can effectively bound (eventually) the number of sets of a given cardinality in any disjoint strong array intersecting A. The other extreme is a set which allows for disjoint strong arrays beating any computable bound.

Definition 6.2 *An ω-immune set A is called* fat *if there is a disjoint strong array $(D_{f(x)})_{x\in\omega}$ intersecting A such that $\#f$ dominates every computable function.*

Playing with quantifiers gives several intermediate degrees of fatness and thinness. We will only show that the two notions defined above are proper. Obviously no set can be fat and thin at the same time.

6.1 The fat sets

In this section we will prove the existence of fat sets in every noncomputable c.e. degree. So far we have not been able to find natural examples of fat sets.

Theorem 6.3 *Every noncomputable c.e. degree contains a fat set.*

As always the theorem follows from a lemma.

Lemma 6.4 *For every noncomputable c.e. set B, there is a fat set $A \leq_{\mathrm{wtt}} B$, such that B disjunctively reduces to A.*

Note that this implies Theorem 3.5 and Lemma 3.6 since fat sets are ω-immune but not h-immune.

Proof. By a finite injury argument we will construct a set A, and a computable function f such that $(D_{f(x)})_{x\in\omega}$ is a strong disjoint array intersecting A. There are three types of requirements. The P requirements code B into A on the even elements, while the R and N requirements guarantee that A is fat if restricted to the odd elements (hence A is fat). The negative N requirements make A ω-immune.

Let $F_e = \{2^{e+1}, 2^{e+1}+2, \ldots, 2^{e+2}-2\}$. Note that $|F_e| = 2^e$ and F_e contains only even numbers. We fulfill

$$P_e \quad : \quad e \in B \text{ iff } A \cap F_e \neq \emptyset,$$

for all e. This will ensure that B disjunctively reduces to A.

To assure that A intersects enough sets in the strong disjoint array we satisfy

$$\begin{aligned} R_{e,k} \quad : \quad & \text{if } \varphi_e(k)\downarrow \text{ and } k \geq e, \text{ then} \\ & (\#f)(k) \geq \varphi_e(k), \end{aligned}$$

for all e, k.

Finally we have to make sure that A is ω-immune by meeting

$$\begin{aligned} N_e \quad : \quad & \text{if } (D_{\varphi_e(n)})_{n\in\omega} \text{ is a disjoint strong } e\text{-array}, \\ & \text{then there is an } n \text{ such that } D_{\varphi_e(n)} \text{ does not intersect } A, \end{aligned}$$

for all e.

The construction is a finite injury argument with regard to the N type of requirements. The priority ordering is

$$\begin{array}{lllllll} & R_{0,0} & & & < P_0 & < N_0 \\ < & R_{0,1} & < R_{1,1} & & < P_1 & < N_1 \\ & & & \vdots & & \\ < & R_{0,k} & < \cdots & < R_{k,k} & < P_k & < N_k \end{array}$$

Requirements of type P and R will be fulfilled instantaneously and never be injured afterwards.

During the construction collections of finite sets of integers will be assigned to requirements of all three types. More exactly F_e (containing even integers) will be assigned to P_e when e enters B (and this will be the only set assigned to P_e). Both R and N type requirements will be assigned more than one set in general. Sets assigned to $R_{e,k}$ will contain k elements. Sets assigned to N_e will contain more than e elements. Furthermore a requirement N_e has associated with it a *taboo set* T_e of at most e elements it tries to keep out of A.

Let g be a computable function enumerating B without repetitions.

Stage $s = 0$. Let $A^0 = \emptyset$, f undefined everywhere, $u^0 = 1$. Initially nothing is assigned to any R_e, and all the taboo sets T_e are empty.

Stage $3s+1$. (Satisfy $P_{g(s)}$.) Let $e = g(s)$. We will satisfy P_e. The set $A^{3s} \cap F_e$ is currently empty, since g enumerates B without repetition, and only P_e can put an element of F_e into A. Consider the set $T = \bigcup_{i=0}^{e-1} T_i$. Then T contains at most $(e-1)e/2 < 2^e = |F_e|$ elements, so we can choose the smallest $a \in F_e - T$ and enumerate a into A, i.e. $A^{3s+1} \cap F_e = \{a\}$, and $A^{3s+1} - F_e = A^{3s}$. Assign F_e to P_e. Let $u^{3s+1} = u^{3s}$. Requirement P_e is satisfied, and will remain so henceforth. Reset all taboo sets of N requirements of lower priority, i.e. $T_i = \emptyset$ for all $i \geq e$.

Stage $3s+2$. (Satisfy R type requirements.) Say that $R_{e,k}$ *requires attention at stage* $3s+2$ if $\varphi_{e,s}(k)\downarrow$ and $k \geq e$ and $R_{e,k}$ has not received attention before. Let $R_{e,k}$ be the highest priority requirement requiring attention. We say that $R_{e,k}$ *receives attention.* Let $\varphi_{e,s}(k) = y$, and $H_i = \{u^{3s+1} + i * 2k, u^{3s+1} + 2 + i * 2k, \ldots, u^{3s+1} + (i+1) * 2k - 2\}$ for $0 \leq i < y$ that is the H_i partition the first $y * k$ odd integers beyond u^{3s+1} into y blocks of length k each. Assign $(H_i)_{0 \leq i < y}$ to requirement $R_{e,k}$. Enumerate the smallest element of each H_i into A^{3s+2}, i.e. $A^{3s+2} \cap H_i = \{u^{3s+1} + i * 2k\}$, and leave the rest of A unchanged. Extend f to include canonical indices of all the H_i $(0 \leq i < y)$ in its range. Let $u^{3s+2} = u^{3s+1} + y * 2k$. Requirement $R_{e,k}$ is satisfied, and will remain so henceforth. Reset all taboo sets of N type requirements of lower priority, i.e. $T_i = \emptyset$ for all $i \geq k$.

Stage $3s+3$. (Satisfy N type requirements, and ensure $A \leq_{\text{wtt}} B$.) Say N_e *requires attention at stage* $3s+3$ if there is no collection of sets assigned to N_e, and there exist t and x such that

(*i*) $\langle e, t, x \rangle < s+1$, and

(*ii*) $|D_{\varphi_{e,t}(x)}| \leq e$, and

(*iii*) $D_{\varphi_{e,t}(x)}$ lies completely to the right of any set assigned to any requirement (P, R or N) of higher priority, i.e. if $z \in D_{\varphi_{e,t}(x)}$ and $w \in H$ where H is a set assigned to a requirement of higher priority, then $z > w$, and

(*iv*) $g(s+1) < \min(D_{\varphi_{e,t}(x)})$.

Choose the smallest e such that N_e requires attention at stage $3s+3$. Say requirement N_e *receives attention at stage* $3s+3$. Fix t and x. Let $T_e = D_{\varphi_{e,t}(x)}$. Remove all assignments of collections of sets to N type requirements of lower priority than N_e, and assign all sets assigned to R or P type requirements that intersect T_e to N_e. Note that condition (*iii*) implies that these sets belong to R or P type requirements of lower priority than N_e, so they contain more than e elements. For every set H that is assigned to N_e, and for which $H \cap A^{3s+3} \neq \emptyset$, let $H \cap A^{3s+3}$ contain only the smallest element of H that is not in T_e (this is possible, since $|H| > e$ and $|T_e| \leq e$). Otherwise A^{3s+3} remains unchanged. Let all $T_{e'} = \emptyset$ for $e' > e$, and $u^{3s+3} = 1 + 2\max\{u^{3s+2}, \max(T_e)\}$. *End of Construction.*

If we assume that A^s converges to a set A, then we see that by construction all P_e are satisfied: if $e \notin B$, then there will never be an element in $A \cap F_e$; if $e \in B$, then an element from F_e is put into A without injuring any higher priority N type requirements as argued during the construction. During stages $3s+3$ this element might be moved within F_e, but this does not affect P_e (assuming that A exists).

Likewise we argue that every $R_{e,k}$ will be satisfied if A^s converges to a set A. (Note that since u^s takes on only odd values the P and R type requirements do not interfere with each other.)

Let H be a set assigned to a P or R type requirement. We have to argue that $H \cap A^s$ converges. Since g enumerates B without repetition $\liminf_{s\to\infty} g(s) = \infty$, so there is a stage after which no taboo set can intersect H (by condition (iv)). So $H \cap A^s$ will remain unchanged from this stage onwards. Since only elements from sets assigned to R or P type requirements are ever enumerated into A this proves that $\lim_{s\to\infty} A^s = A$ exists.

So all P and R requirements will be fulfilled. Furthermore note that every P and R requirement acts at most once.

Suppose that not all requirements of type N are satisfied. Choose the N_e of highest priority which is not satisfied. Requirements of type N can be injured by all other types of requirements. Since P and R requirements act at most once it can be proved by induction that every N type requirement is injured at most finitely often, hence there is a stage s' after which no P or R type requirement of higher priority than N_e will act, and no N_i with $i < e$ will require attention. In particular the sets assigned to requirements of higher priority than N_e will not change after stage s'.

There are no sets assigned to N_e at stage s' and $(D_{\varphi_e(x)})_{x\in\omega}$ is a disjoint strong e-array (otherwise N_e would be satisfied, and remain satisfied).

Given y we can effectively find x and t such that conditions (ii) and (iii) of the construction are satisfied, and moreover $\min(D_{\varphi_e(x)}) > y$. Thus at stage $s = \max\{s', \langle e, x, t\rangle\}$ all conditions except (iv) are satisfied. Since N_e cannot receive attention (otherwise it would be satisfied permanently), we can conclude that $y < \min(D_{\varphi_e(x)}) \leq g(s''+1)$ for all $s'' > s$. Hence $y \in B$ iff $y \in \{g(0), \ldots, g(s)\}$ which is impossible, since B is not computable. Therefore N_e is satisfied.

We are left with the proof that $A \leq_{\mathrm{wtt}} B$. But this follows from the permitting in condition (iv) together with (iii): given y simulate the construction of A until in some stage $3s+2$ a set H which lies completely to the right of y is assigned to a requirement $R_{e,k}$, i.e. if h is the smallest element of H, then $y < h$. Use B to determine a stage $t > 3s+2$ such that $\{g(0), \ldots, g(t)\} = B{\restriction}h$. Since g enumerates B without repetitions, from this stage on conditions (iii) and (iv) ensure that A does not change below h, so we can decide whether $y \in A$ by continuing to simulate the construction of A up to stage t. This gives us a wtt-reduction from A to B. $\square$

6.2 The thin sets

At the other end of the spectrum we have thin sets. Thin sets occur more naturally than fat sets, and instead of another tedious degree construction we give two very different examples of thin sets. Both examples will fulfill a stronger property than thinness: rather than one computable function that dominates every counting function eventually, there will be an algorithm that finds an (everywhere) dominating function uniformly for every strong disjoint array. To make this precise:

Definition 6.5 *A set A is* uniformly thin *if there is a total computable function f such that $(\#\varphi_e)(n) \leq f(e,n)$ for every strong disjoint array $(D_{\varphi_e(x)})_{x\in\omega}$ intersecting A.*

Lemma 6.6 *Every uniformly thin set is thin.*

Proof. Suppose A is uniformly thin via f. Let $d(n) = \max\{f(e,n) : e \le n\}$. If φ_e describes a disjoint strong array, then $(\#\varphi_e)(n) \le d(n)$ for every $n \ge e$. $\square$

Sets which are c.e. and have a uniformly thin complement turn out to be good old friends.

Definition 6.7 *A set A is* effectively simple *if it is c.e. and there is a computable function f such that*

$$W_e \subset \overline{A} \Longrightarrow |W_e| < f(e).$$

Theorem 6.8 *The complement of an effectively simple set is (uniformly) thin.*

Before proving the theorem we note several consequences. First of all the theorem gives us a host of natural examples of (uniformly) thin sets like the complement of Post's simple set, and R. Secondly it yields an alternative characterization of effectively simple sets.

Corollary 6.9 *A c.e. set is effectively simple if and only if its complement is uniformly thin.*

One direction is covered by the theorem, the other is trivial. The corollary might be helpful in separating the notions of uniformly thin and thin. All that would be necessary is to construct a simple set which is not effectively simple, and whose complement is thin.

The theorem is a strong effective version of the folklore result we presented in the beginning (Lemma 3.2). The proof presented there can be recast to show that it really depends on the non-uniform knowledge of two parameters $m = \limsup_{x\to\infty} |D_{g(x)} \cap \overline{A}|$ and the point n such that $|D_{g(x)} \cap \overline{A}| \le m$ for $x \ge n$. These are serious obstacles for an effectivization, and it takes some work to overcome them.

Proof of Theorem 6.8. Let $\overline{A}$ be effectively simple, hence $\overline{A}$ is c.e. and there is a computable function f such that $W_e \subset A \Longrightarrow |W_e| < f(e)$. We write $\overline{A}_s$ for an effective approximation to $\overline{A}$.

Call g a *partial disjoint strong n-array for A* if

(*i*) $D_{g(x)} \cap D_{g(y)} = \emptyset$ for all $x, y \in \mathrm{dom}(g)$, and

(*ii*) $|D_{g(x)}| = n$ for all $x \in \mathrm{dom}(g)$,

(*iii*) $D_{g(x)} \cap A \ne \emptyset$ for all $x \in \mathrm{dom}(g)$.

We claim that there is a computable (total) function h such that $|\mathrm{dom}(g)| < h(e,n)$ for every $g = \varphi_e$ which is a partial disjoint strong n-array for A. With this claim we can finish the proof of the theorem. Define a computable function $k(e,n)$ by

$$\varphi_{k(e,n)}(x) = \begin{cases} g(x) & \text{if } |D_{\varphi_e(x)}| = n, \\ \uparrow & \text{else,} \end{cases}$$

and let $f(e,n) = h(k(e,n),n)$. Suppose $g = \varphi_e$ describes a strong disjoint array intersecting A. Then $(\#g)(n) = |\mathrm{dom}(k(e,n)| < h(k(e,n),n) = f(e,n)$, so f witnesses that A is uniformly thin, hence it is thin.

For the proof of the claim assume that $g = \varphi_e$ is a partial disjoint strong n-array for A. (In the following note that even if φ_e is not, the algorithm for $h(e,n)$ will compute some (meaningless) value.)

To compute $h(e,n)$ we will use an intermediate function $h'(e,n,m)$ defined by induction on m. The idea is that $h'(e,n,m)$ computes a correct upper bound on $|\mathrm{dom}(g)|$ if we know that $\max\{|D_{g(x)} \cap \overline{A}| : x \in \mathrm{dom}(g)\} \leq m$.

It is easy to compute $h'(e,n,0)$ because we can assume that $D_{g(x)} \subset A$ for all $x \in \mathrm{dom}(g)$. So using f we can effectively determine an upper bound on $|\mathrm{dom}(g)|$. Let us show how to compute $h'(e,n,m)$ inductively from $h'(\cdot,n,m-1)$.

We know that $\max\{|D_{g(x)} \cap \overline{A}| : x \in \mathrm{dom}(g)\} \leq m$. Consider the sets $B = \{y_{m+1},\ldots,y_n : (\exists x)[D_{g(x)} = \{y_1,\ldots,y_n\}, y_1,\ldots,y_m \in \overline{A}]\}$ and $C = \{x : x \in \mathrm{dom}(g), |D_{g(x)} \cap \overline{A}| \geq m\}$. Since C contains the indices for the elements collected in B we know that $|C| \leq |B|$. Now B is a c.e. subset of A, and we can find an index for it effectively in n and uniformly in g. Using f we get an upper bound c on the cardinality of B and hence we know that $|C| \leq c$.

Compute a (partial) function $r(i)$ as follows: search for the smallest $\langle x_1,\ldots,x_i,s\rangle$ for which $x_1,\ldots,x_i$ are pairwise different, and $D_{\varphi_{e,s}(x_j)} \cap \overline{A}_s$ contains at least m elements for every $1 \leq j \leq i$. Let $r(i)$ be $\langle x_1,\ldots,x_i\rangle$.

The computation of $r(|C|)$ converges and computes the indices of those sets which have exactly m elements in $\overline{A}$ (remember that we are assuming that $\max\{|D_{g(x)} \cap \overline{A}| : x \in \mathrm{dom}(g)\} \leq m$). If x is different from all $|C|$ indices coded by $r(|C|)$, then $|D_{g(x)} \cap \overline{A}| \leq m-1$. Thus if we knew $|C|$ we could compute h' recursively. Unfortunately we do not know the correct value of $|C|$, so we have to consider the c functions ($0 \leq i < c$):

$$\varphi_{e(i)}(x) = \begin{cases} g(x) & \text{if } x \neq x_j \text{ for all } 1 \leq j \leq i, \\ & \text{where } r(i) \downarrow = \langle x_1,\ldots,x_i\rangle \\ \uparrow & \text{else,} \end{cases}$$

Every such $\varphi_{e(i)}$ describes a partial disjoint strong n-array for A, and one of them (namely $\varphi_{e(|C|)}$) fulfills the inductive condition that all intersections with $\overline{A}$ have cardinality less than $m-1$. Hence we will get an upper bound on $|\mathrm{dom}(g)|$ if we consider all possibilities, i.e. let $h'(e,n,m) = \max_{0 \leq i < c}\{i + h'(e(i),n,m-1)\}$. Finally defining $h(e,n) = h'(e,n,n-1)$ finishes the proof of the claim. $\square$

There is one more natural example for a thin set we should mention.

Theorem 6.10 MIN *is (uniformly) thin.*

We leave the proof to the reader. The basic observation is that MIN is effectively k-immune uniformly in k.

7 Cuppability

Post showed that a simple set cannot be btt-complete. We have generalized that result to non-c.e. sets by isolating the immunity property which is responsible for the incompleteness.

Another approach to the btt-incompleteness of simple sets would have been through degrees. How incomplete are simple sets? Putting it differently: can the join of a btt-incomplete degree with a simple degree be btt-complete? We will show that the answer is no.

Definition 7.1 *A set A is called* r-cuppable*, if there is a c.e. set B such that $\emptyset' \leq_{\mathrm{r}} A \oplus B$ and $\emptyset' \not\leq_{\mathrm{r}} B$, where r is a class of reductions (like m, 1, btt, c, d, tt, wtt, T, e)*

The following three results are known:

Theorem 7.2

- *An immune set is not m-cuppable* (Lachlan [10]).
- *A hypersimple set is not wtt-cuppable* (Downey, Jockusch [4]).
- *A $\emptyset'$-hypersimple set is not e-cuppable* (Nies, Sorbi [12]).

We will now show that an analogous statement is true for btt-reductions as well.

Theorem 7.3 *A simple set is not btt-cuppable.*

For the proof we need to have a closer look at some older results.

Proposition 7.4 (Lachlan [10]) *If $\emptyset' \leq_{\mathrm{m}} A \times B$ and at least one of A and B is c.e., then either $\emptyset' \leq_{\mathrm{m}} A$ or $\emptyset' \leq_{\mathrm{m}} B$.*

Note that the proposition implies that immune sets are not m-cuppable. Kobzev proved the following variant of Lachlan's result by making some slight modifications to the proof.

Proposition 7.5 (Kobzev [8])

(*i*) *If A is productive and B is c.e., then either $A \cap B$ or $A \cap \overline{B}$ is productive.*

(*ii*) *If A is creative and B is computable, then either $A \cap B$ or $A \cap \overline{B}$ is creative.*

The second item easily follows from the first. For the theorem we need a uniform version of the second item. Lachlan's result, however, uses a nonuniform proof: two strategies are pursued, one trying to build a reduction to A and the other a reduction to B. Furthermore the second strategy will only yield a reduction which is correct up to finitely many errors. Fortunately the first strategy yields a reduction uniformly, hence if we know that the first strategy succeeds we get the reduction uniformly.

Lemma 7.6 *If A is creative and B is computable and we know that exactly one of $A \cap B$ and $A \cap \overline{B}$ is creative and which one it is, then we can find a reduction from $\emptyset'$ to that set uniformly in A and B.*

Proof. Assume that $\emptyset' \leq_{\mathrm{m}} A$ and B is computable. We can then (uniformly) find a reduction from $\emptyset'$ to $(A \cap B) \times (A \cap \overline{B})$ (using that A is c.e. and not empty). Assume that exactly one of $(A \cap B)$ and $A \cap \overline{B}$ is creative and we know which one. Start the Lachlan proof with the uniform strategy working on the set we know to be creative. Since the other strategy has to fail (the other set not being creative) this will (uniformly) yield a reduction to the creative set. □

We need another result for the proof. This result was proven independently by Lachlan and Kobzev, the former using the technique of Proposition 7.4, the latter giving a proof using his result on productive sets.

Proposition 7.7 (Kobzev [8], Lachlan [13]) *A btt-complete set is bd-complete.*

With this we are now ready to prove our result.

Proof of Theorem 7.3. Suppose $\emptyset' \leq_{\mathrm{btt}} A \oplus B$ where A is simple, and B is c.e. By Proposition 7.7 we know that there are two computable functions f and g from ω to $\mathcal{P}(\omega)$ such that $x \in K$ iff $f(x) \cap A \neq \emptyset$ or $g(x) \cap B \neq \emptyset$.

For a set D define $E_D = \{x : f(x) \subseteq D\}$. We claim that there is a finite set $D \subseteq \overline{A}$ such that $\emptyset' \cap E_D$ is creative. Since $x \in \emptyset' \cap E_D$ iff $x \in E_D$ and $g(x) \cap B \neq \emptyset$ the claim implies that $\emptyset' \leq_{\mathrm{m}} \emptyset' \cap E_D \leq_{\mathrm{bd}} B$ and we are done.

We are left with the verification of the claim. Assume for a contradiction that $\emptyset' \cap E_D$ is not creative for any finite set $D \subseteq \overline{A}$. Because of Proposition 7.5 this means that $\emptyset' \cap \overline{E_D}$ is creative for all finite subsets D of $\overline{A}$. By Lemma 7.6 we can even find a productive function for $\overline{\emptyset' \cap \overline{E_D}}$ uniformly in the finite set D. Start with $D_0 = \emptyset$ and $C_0 = E_{D_0}$. Then C_0 is a c.e. set in the complement of $\emptyset' \cap \overline{E_{D_0}}$, hence we can find an element $y_0 \in \overline{K} \cap \overline{E_{D_0}}$ using the productive function. For this element we have $f(y_0) \cap A = \emptyset$ and $f(y_0) \not\subseteq D_0 = \emptyset$. Repeat this procedure with $D_1 = f(y_0)$, $C_1 = E_{D_1}$, and so on. Because of the uniformity we get a c.e. set $\bigcup_{i=0}^{\infty} D_i$ which is a subset of $\overline{A}$, since all the D_i are, and infinite, since $D_{n+1} \not\subseteq D_n$. This contradicts the simplicity of A. □

A closer look at the proof shows that we have also proved the following theorem.

Theorem 7.8 *A simple set is not d-cuppable.*

This is as much as we can expect from a simple set, since simple sets can be c-complete.

8 Conclusion

We have shown that k-immunity is a powerful tool in the study of bounded-truth table reductions and establishes some hitherto unknown connections (between retraceability and btt-hardness for example). We believe that by defining k-immunity explicitly we have isolated a helpful notion that has been used implicitly in results of the past. Further research could aim at collecting and unifying some of these results. It seems, for example, that k-immunity might play a role in the study of effective versions of Ramsey's Theorem or in frequency computation.

We also provided some initial results on ω-immune sets which are not hyperimmune, and showed that there is a variety of sets residing in this realm, in particular the thin sets which generalize effectively simple sets.

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