Computability of Convex Sets *

Martin Kummer
Institut für Logik, Komplexität
und Deduktionssysteme
Universität Karlsruhe
D-76128 Karlsruhe, Germany
kummer@ira.uka.de

Marcus Schäfer
Department of Computer Science
University of Chicago
1100 East 58th Street
Chicago, Illinois 60637, USA
schaefer@cs.uchicago.edu

Abstract

We investigate computability of convex sets restricted to rational inputs. Several quite different algorithmic characterizations are presented and proved to be equivalent, like the existence of effective approximations by polygons or effective line intersection tests. Also obtained are characterizations of recursively enumerable convex sets.

1 Introduction

Convex sets play a prominent role in mathematical programming, computational geometry, convex analysis, and many other areas. There is a large number of papers dealing with polynomial time computable convex sets, but the basic question "Which convex sets are computable?" was scarcely studied.

In this paper we characterize computable convex sets and show that several quite different approaches lead to the same notion. We consider computations on rational inputs, as e.g. in [4]. This has the advantage that the standard notions of computability and complexity on the natural numbers can be applied.

We call a convex set A computable if there is an algorithm which computes the characteristic function of A for all rational inputs that do not belong to the boundary of A. It turns out that all other apparently weaker notions which have been proposed in the literature are equivalent to this definition. For instance, the existence of an algorithm for the "weak membership problem" of Grötschel, Lovász, and Schrijver [4], or the property of being "Turing located" introduced by Ge and Nerode [3].

On the other hand, the apparently stronger requirement that there is a computable double sequence of rational polygons converging to A from inside and from outside, is also equivalent. Finally, if it is decidable which rational lines intersect the interior of A, then A is already computable.

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As an application we give a new proof of a result of Ge and Nerode, that the extreme points of a convex set can be approximated effectively. We also obtain several characterizations of recursively enumerable convex sets.

For the sake of exposition we restrict ourselves to convex regions in $\mathbb{R}^2$, but with more technical and notational effort it is possible to generalize the definitions and results to higher dimensions.

Most parts of Section 5 originally appeared in [10].

2 Some Useful Facts about Convex Sets

Before switching to the *computable* viewpoint, it will be helpful to review some purely geometric results, which will be used later.

2.1 Notation

We will deal with a special class of convex sets, namely, compact convex sets in $\mathbb{R}^2$ which have an interior point. We will call these sets *convex regions.*

The following geometric notations are used. We write xy for the line segment joining the points x and y. The line through these points is denoted by $l(x, y)$. The convex hull of a set A is denoted by $conv(A)$. As usual $S(x, r)$ is the closed circle (sphere) with radius r and center x. The set of *interior* points of A is denoted by $int(A)$. The exterior points $ex(A)$ are the interior points of the complement A^c of A. The boundary of A is written as ∂A. In some places we need the closure $cl(A)$ of A. The distance between two points x, y is denoted by $d(x, y)$; the distance from a point x to a set A is denoted by $d(x, A) := \inf_{y \in A} d(x, y)$.

2.2 The Hausdorff distance

We will need a notion of what it means for two convex sets not to differ too much. The standard tool in this respect is the Hausdorff distance $d_H(A, B)$ which for two subsets A and B of $\mathbb{R}^2$ is defined as

$$d_H(A, B) := \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

On compact convex subsets of $\mathbb{R}^2$ the Hausdorff distance is a metric.

We assemble some facts which are easily proven (proofs of the first two facts can be found in [10], the third is trivial).

The first fact allows us to compute the Hausdorff distance between a convex set and a surrounding polygon.

Fact 2.1 *Let A be a convex region contained in the convex hull $B := conv\{b_1, \ldots, b_n\}$ of n points, then $d_H(A, B) = \max\{d(b_i, A) : 1 \leq i \leq n\}$.*

The second fact gives an upper bound for the Hausdorff distance between two convex polygons.

Fact 2.2 $d_H(conv\{a_1, \ldots, a_k\}, conv\{b_1, \ldots, b_l\}) \leq d_H(\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_l\})$.

The third fact gives a useful upper bound on the Hausdorff distance of two sets.

Fact 2.3 *If $B \subseteq C \subseteq D$ then $d_H(E, C) \leq \max\{d_H(E, B), d_H(E, D)\}$.*

We now present a more interesting result that holds if two convex sets are close (with regard to d_H).

Definition 2.4 [4] *Let A be a subset of $\mathbb{R}^2$ and $\delta > 0$. Define*

$$A_{-\delta} := \{x \in A : S(x, \delta) \subseteq A\}.$$

$A_{-\delta}$ is called the interior δ-ball *of A.*

Lemma 2.5 *Let A and B be convex regions such that $d_H(A, B) < \delta$. Then $A_{-\delta}$ is a subset of B.*

Proof Suppose there is a point x with $S(x, \delta) \subseteq A$, but $x \notin B$. Then there exists a point $y \in B$ with minimal distance from x, since B is compact. For the same reason the distance is positive and hence $x \neq y$. Since y has minimal distance this distance is bounded by $d_H(A, B)$ and is therefore less than δ. This implies $y \in S(x, \delta) \subseteq A$. So we can define a point w as the intersection of a ray from y through x with the boundary of A. We claim that there is no point of B closer to w than y, that is $d(w, B) = d(w, y)$. If this holds, the contradiction $d_H(B, A) \geq d(w, B) = d(w, y) > d(w, x) = \delta$ follows. To prove the claim assume there is a point $z \in B$ with $d(z, w) < d(w, y)$. Then zy intersects $S(x, d(x, y))$ and thus contains an interior point of $S(x, d(x, y))$. As the whole of zy lies in B, this point of B would be closer to x than y which is impossible. ▮

2.3 Extreme Points

The notion of extreme points from convex geometry will prove helpful with regard to line intersection tests which we consider later.

Definition 2.6 (Extreme points) *A point x is said to be* extreme *for a convex set A, if it belongs to A but does not lie on a line strictly between two other points of A. The set of extreme points of A is denoted by $ext(A)$.*

We will only consider the special case in which A is a convex region.

Fact 2.7 *If A is a convex region, then $ext(A)$ is compact.*

This fact does not hold for higher dimensions. Still our results and proofs can be generalized if $ext(A)$ is replaced by $cl(ext(A))$.

The most important result on extreme points is by Minkowski (later generalized as the famous Krein-Milman theorem of functional analysis):

Theorem 2.8 (Minkowski [2, 7]) *Let A be a convex region. Then A is the convex hull of its extreme points:*

$$A = conv(ext(A)).$$

We will need the following, stronger version of this theorem. For the sake of completeness we include a proof.

Theorem 2.9 *Let A be a convex region. For every $\varepsilon > 0$ there is a finite subset D of $ext(A)$ such that $d_H(conv(D), A) < \varepsilon$.*

To prove this we will first consider dense subsets of $ext(A)$ and use the fact that there are dense and countable subsets of $ext(A)$ to get the connection to the finite case. The Theorem of Minkowski immediately yields the following lemma.

Lemma 2.10 *Let A be a convex region and C be a dense subset of $ext(A)$. Then $d_H(A, conv(C)) = 0$.*

Proof Since C is a dense subset of the compact set $ext(A)$ we have $cl(C) = ext(A)$ and therefore

$$A \subseteq conv(cl(C)) = cl(conv(C)) \subseteq cl(A) = A$$

which implies $d_H(A, conv(C)) = 0$. ∎

The next lemma gives us the connection between countable sets and the finite case.

Lemma 2.11 *Let $(D_n)_{n\in\mathbb{N}}$ be an ascending sequence of sets in $\mathbb{R}^2$ such that $D := \bigcup_{n\in\mathbb{N}} D_n$ is bounded. Then $d_H(D_n, D)$ converges to zero as $n \to \infty$.*

Proof Set $d_n := d_H(D_n, D)$. Then $(d_n)_{n\in\mathbb{N}}$ is a descending sequence. If this sequence does not converge to zero it must converge to an $\varepsilon > 0$ and $d_n \geq \varepsilon$ for all n. But $d_H(D_n, D) \geq \varepsilon$ implies that there is a point $x_n \in cl(D)$ with $d(x_n, D_n) \geq \varepsilon$. Since $cl(D)$ is bounded it is compact and the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to a point $x \in cl(D)$. Choose m such that $d(x_n, x) \leq \frac{\varepsilon}{2}$ for all $n \geq m$. Then $d(x, D_{n_k}) \geq d(x_{n_k}, D_{n_k}) - d(x, x_{n_k}) = \frac{\varepsilon}{2}$ for all $n_k \geq m$. This contradicts the fact that every neighbourhood of x must contain points of almost all D_n. ∎

Proof of Theorem 2.9 Fact 2.7 tells us that $ext(A)$ is compact. The Heine-Borel characterization of compact sets implies that $ext(A)$ contains a countable and dense subset $C = \{x_1, x_2, \ldots\}$. Let $D_n := conv\{x_1, \ldots, x_n\}$. We apply the last lemma with $D = conv(C)$ to obtain that $d_H(D_n, conv(C))$ converges to zero. Since $d_H(conv(C), A) = 0$ we have that $d_H(D_n, A)$ converges to zero, which proves the theorem. ∎

At the core of Minkowski's Theorem is the following fact, which will be useful later.

Fact 2.12 *If A is a convex region and H a halfspace for which $H \cap A$ is nonempty, then $H \cap A$ contains an extreme point of A.*

The definition of an extreme point is not very helpful in terms of localizing such a point, since the characterization is negative. As a remedy we present a lemma.

Lemma 2.13 *Let A be a convex region. Then $x \in ext(A)$ if and only if $x \in \partial A$ and for every $\delta > 0$ there is a rational halfspace H containing x, for which $H \cap A \subseteq S(x, \delta)$.*

By rational halfspace H we mean a closed halfspace in $\mathbb{R}^2$ bounded by a line that passes through two distinct rational points.

Proof $(\Leftarrow)$: By assumption every neighbourhood of x contains points of both A and A^c which makes x a boundary point of A. For sake of contradiction suppose that x is not extreme. That is there are two points y and z of A different from x with $x \in yz$. Choose $\delta > 0$ such that y and z lie outside of $S(x, \delta)$. But any halfspace containing x must either contain y or z contradicting $H \cap A \subseteq S(x, \delta)$.

$(\Rightarrow)$: Suppose there is a $\delta > 0$ such that any rational halfspace containing x violates $H \cap A \subseteq S(x, \delta)$. Put

$$T := \{w \in \partial S(x, \delta) : (\exists w' \in A)[w \in w'x]\}.$$

The set T is compact, since A is. We claim that any rational halfspace containing T must already contain x. Suppose to the contrary that there is a rational halfspace K with $T \subseteq K$ and $x \notin K$. By definition of T the rational halfspace $H := cl(K^c)$ fulfills $H \cap A \subseteq S(x, \delta)$ and since $x \notin K$ certainly $x \in H$ contradicting that no such halfspace exists.

As a convex hull of a compact set is the intersection of all halfspaces containing the set and this remains true if we confine the intersection to rational halfspaces, we have that $x \in conv(T)$ (using the claim). As x does obviously not belong to T this constitutes a contradiction to the extremality of x. ▮

3 The Weak Characteristic Function

For background on computability the reader is referred to [8]. The domain of a partial function f is denoted by $dom(f)$, we write $f(x) = \uparrow$ if $x \notin dom(f)$. A partial function $f : \mathbb{N}^k \to \mathbb{N}$ is computable if there is a program which terminates for input x iff $x \in dom(f)$ and then outputs $f(x)$. We identify $\mathbb{N}$ and $\mathbb{Q}$ by some standard (computable) bijection $\pi : \mathbb{N} \to \mathbb{Q}$. In particular, a partial function $f : \mathbb{Q} \to \mathbb{Q}$ is computable iff the corresponding function $\pi^{-1} \circ f \circ \pi$ is computable over $\mathbb{N}$.

We are interested in establishing a suitable notion of computability for convex sets. As we have decided to use the rational model for our investigations the following definition might appear natural: a set A is called computable if we can effectively decide whether a rational point lies in A or not. Approaching convex sets in a geometric setting will show that this definition does not lead very far. Typically geometric operations depend on interior points and cannot distinguish two sets differing only on their boundary. Another problem is that the boundary of a set may turn out to be quite complicated (in the sense of computability) although the set itself and its complement are not: take for example the circle around the origin with radius 1. There are uncountably many sets that differ from this circle only in the rational points on the boundary, so most of them are not computable.

This motivates the following definition which will prove invariant to different approaches.

Definition 3.1 (Weak characteristic function) *Let A be a subset of $\mathbb{R}^2$. The function*

$$\omega_A : \mathbb{Q}^2 \to \{0,1\} : x \mapsto \begin{cases} 1 & x \in int(A) \\ 0 & x \in ex(A) \\ \uparrow & else \end{cases}$$

is called the weak characteristic function *of A.*

The attribute "weak" is justified for convex regions as in this case the weak characteristic function is Turing reducible to the characteristic function (even uniformly). The converse is not true as is shown by our next result.

Theorem 3.2 *There is a convex region A such that ω_A is partial recursive but no total extension of ω_A is (in particular, neither χ_A nor $\chi_{int(A)}$ are computable, when restricted to $\mathbb{Q}^2$).*

Proof Let (B, C) be a pair of recursively inseparable recursively enumerable sets, i.e., B, C are disjoint recursively enumerable subsets of $\mathbb{N}$ and there is no total computable function $f : \mathbb{N} \to \{0,1\}$ such that $f(x) = 1$ for all $x \in B$ and $f(x) = 0$ for all $x \in C$. See [8, p. 148 f.] for the existence of such sets.

One starts with a set S which is the convex hull of infinitely many rational points $p_0, p_1, \ldots$ which are vertices of S. The set A will be constructed such that all of $p_{2n}, n \in \mathbb{N}$ are vertices of A and if $n \in B$ then $p_{2n+1} \in int(A)$ and if $n \in C$ then $p_{2n+1} \in ex(A)$. To achieve this we suitably replace some of the vertices p_{2n+1} by new ones in such a way that $\omega(A)$ is still computable. There can be no computable total extension g of ω_A, since otherwise $f(n) = g(p_{2n+1})$ would witness that (B, C) are not recursively inseparable.

More formally, we choose a sequence $(p_i)_{i \in \mathbb{N}}$ of rational points with the following properties:

- (i) $conv\{p_0, p_1, \ldots\}$ has vertices $p_0, p_1, \ldots$ in clockwise order. $p_0 = (-1, 0)$ and $(p_i)_{i \in \mathbb{N}}$ converges to $p_\infty = (1, 0)$
- (ii) The sequence $(p_i)_{i \in \mathbb{N}}$ is effectively computable.
- (iii) For each p_{2i+1} we can compute a radius $r(i)$ such that if every p_{2i+1} is substituted by some point $p'_{2i+1} \in S(p_{2i+1}, r(i))$, the set $conv\{p_0, p'_1, p_2, p'_3, \ldots\}$ still has vertices $p_0, p'_1, p_2, \ldots$ in this order.

A possible choice is shown in Figure 1.

Let $(q_s)_{s \in \mathbb{N}}$ be an effective enumeration of $\mathbb{Q}^2$ and let $(i_s)_{s \in \mathbb{N}}$ be an effective one-to-one enumeration of $B \cup C$. We define p'_{2i+1} as follows:
If $i \notin B \cup C$, then $p'_{2i+1} = p_{2i+1}$.
If $i \in B \cup C$ there is a unique s such that $i_s = i$. Search a rational point $q \in S(p_{2i+1}, r(i))$ such that:

(1) $q_t \in \triangle(p_{2i}, p_{2i+1}, p_{2i+2}) \;\Rightarrow\; q_t \in \triangle(p_{2i}, q, p_{2i+2})$, for all $t \leq s$.

(2) $q_t \in ex(\triangle(p_{2i}, p_{2i+1}, p_{2i+2})) \;\Rightarrow\; q_t \in ex(\triangle(p_{2i}, q, p_{2i+2}))$, for all $t \leq s$.

(3) $i \in B \;\Rightarrow\; q \in \triangle(p_{2i}, q, p_{2i+1})$.

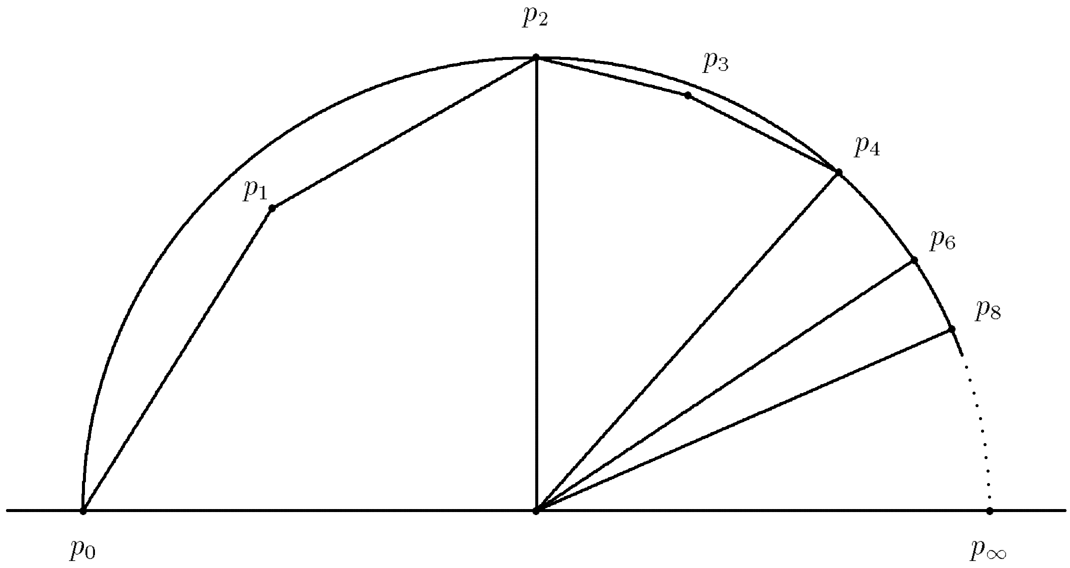

Figure 1: Starting point of the construction

(4) $i \in C \;\Rightarrow\; q \in ex(\triangle(p_{2i}, q, p_{2i+1}))$.

Here $\triangle(a, b, c)$ denotes the interior of the triangle with vertices a, b, c. It is easy to see that the search terminates. Let $p'_{2i+1} = q$.

Now we define $A = conv\{p_0, p'_1, p_2, p'_3, \ldots\}$. By the remarks above it is easily verified that there is no total computable extension of ω_A.

It remains to show that ω_A is computable. To simplify notation let $p^s_{2i+1} = p'_{2i+1}$ if there is $t \le s$ such that $i_t = i$ and $p^s_{2i+1} = p_{2i+1}$ otherwise. Clearly the function $h(s, i) = p^s_{2i+1}$ is computable.

Let $q = (a, b) \in \mathbb{Q}^2$ be given. For $b \le 0$ if $q \in p_0 p_\infty$ then $\omega_A(q) =\uparrow$, else $\omega_A(q) = 0$. Otherwise we compute t and i such that $q = q_t$ and the line segment $q(0, 0)$ lies between $p_{2i}(0, 0)$ and $p_{2i+2}(0, 0)$. By construction, if there is some $s \ge t$ such that $q \in \triangle(p_{2i}, p^s_{2i+1}, p_{2i+2})$, then $\omega_A(q) = 1$; if for some $s \ge t$ we have $q \in ex(\triangle(p_{2i}, p^s_{2i+1}, p_{2i+2}))$, then $\omega_A(q) = 0$. Otherwise $\omega_A(q) =\uparrow$. Thus ω_A is computable by exhaustive search. ∎

4 Some Notions of Approximative Computability

We will now list a couple of other natural algorithmic problems. First we consider two operations on sets which can be regarded as approximations (from a geometric viewpoint).

Definition 4.1 (Weak Membership Tests) *For a set $A \subseteq \mathbb{R}^2$ every partial function $\alpha : \mathbb{Q}^2 \times \mathbb{Q}_+ \to \{0, 1\}$ which for all $x \in \mathbb{Q}^2$ and $r \in \mathbb{Q}_+$ satisfies the conditions*

- $S(x, r) \subseteq int(A) \Rightarrow \alpha(x, r) = 1$
- $S(x, r) \subseteq ex(A) \Rightarrow \alpha(x, r) = 0$

is called a Weak Membership Test (WMT) *for A. We will write α_A for a WMT for A.*

The name WMT is explained by the observation that $\alpha_A(x,r)=1$ implies $d(x,A)\le r$, and $\alpha_A(x,r)=0$ implies $d(x,A^c)\le r$. Note, however, that we do not require α to be everywhere defined. If $S(x,r)$ intersects both A and its complement then $\alpha_A(x,r)$ may give an arbitrary answer or may not terminate at all.

The following definition is due to Grötschel, Lovász, and Schrijver [4].

Definition 4.2 (WMEM) *For a set $A\subseteq\mathbb{R}^2$ we say that* the Weak Membership Problem (WMEM) is solvable *if there is a total recursive function β_A from $\mathbb{Q}^2\times\mathbb{Q}_+$ to $\{0,1\}$ such that*

- $\beta_A(x,r)=1\Rightarrow S(x,r)\not\subseteq ex(A)$
- $\beta_A(x,r)=0\Rightarrow S(x,r)\not\subseteq int(A)$

Analogically to WMTs we investigate another operation which recognizes the lines intersecting the set A and the lines avoiding the set A.

Definition 4.3 (Line Intersection Tests) *Given a set $A\subseteq\mathbb{R}^2$ every partial function $\gamma:\mathbb{Q}^2\times\mathbb{Q}^2\to\{0,1\}$ which satisfies the conditions*

- $l(x,y)\cap int(A)\neq\emptyset\Rightarrow\gamma(x,y)=1$
- $l(x,y)\subseteq ex(A)\Rightarrow\gamma(x,y)=0$

is called a Line Intersection Test (LIT). *We will write γ_A for a LIT for A.*

To define the next notion we need to recall a concept from recursive analysis [5, 9]. A real number $x\in\mathbb{R}^k$ is represented by a function $\phi:N\to\mathbb{Q}^k$ if $|\phi(n)-x|\le 2^{-n}$ for all n. A real function $f:\mathbb{R}^2\to\mathbb{R}$ is computable if there is an oracle Turing machine M such that for every x and every representation ϕ of x, $\lambda n.\ M^\phi(n)$ is a representation of $f(x)$, i.e., M computes a representation of $f(x)$ if it is given a representation of x as an oracle.

Definition 4.4 (Turing located) *A set $A\subseteq\mathbb{R}^2$ is called* Turing located *if the real function $x\mapsto d(x,A)$ is computable.*

5 A Characterization of Computable Convex Sets

We can now state our main theorem which says that for convex regions all the definitions of Sections 4 and 5 are equivalent.

Theorem 5.1 *Let A be a convex region. Then the following are equivalent:*

(i) *ω_A is partial recursive.*

(ii) *There are computable sequences $(P_i)_{i\in\mathbb{N}}$ and $(Q_i)_{i\in\mathbb{N}}$ of rational polygons such that $P_i\subseteq A\subseteq Q_i$ and $d_H(P_i,Q_i)<2^{-i}$ for all i.*

(iii) *There is a partial recursive WMT α_A.*

(iv) *There is a partial recursive LIT γ_A.*

(v) *The Weak Membership Problem for A is solvable.*

(vi) *A is Turing located.*

Proof For the proof we introduce the following variants of (ii), (iii) and (iv):

(ii') There is a computable sequence $(P_i)_{i\in\mathbb{N}}$ of rational polygons such that $d_H(P_i, A) < 2^{-i}$ for all i.

(iii') The WMT $\qquad \alpha_A(x,r) := \begin{cases} 1 & S(x,r) \subseteq int(A) \\ 0 & S(x,r) \subseteq ex(A) \\ \uparrow & \text{else} \end{cases} \qquad$ is partial recursive.

(iv') The LIT $\qquad \gamma_A(x,y) := \begin{cases} 1 & l(x,y) \cap int(A) \neq \emptyset \\ 0 & l(x,y) \subseteq ex(A) \\ \uparrow & \text{else} \end{cases} \qquad$ is partial recursive.

We give an outline of the proof:

$$\begin{tikzcd}
 & & (iv') \arrow[r] & (iv) \arrow[dr] & & & & \\
(i) \arrow[r] & (ii) \arrow[ur] \arrow[dr] & & & (ii') \arrow[r] & (vi) \arrow[dr] & & (i) \\
 & & (iii') \arrow[rrrr] & & & & (iii) \arrow[ur] \arrow[r, leftrightarrow] & (v)
\end{tikzcd}$$

$(i) \Rightarrow (ii)$ The existence of rational polygons as in (ii) is guaranteed by the well-known Lemma of Hadwiger (see [6]). To prove $(i) \Rightarrow (ii)$ we need an effective version of this Lemma. The idea for this is to compute ω_A on a sufficiently fine rational grid which covers A.

Choose a rational point p, and $a, b \in \mathbb{Q}_+$ such that $S(p,a) \subseteq A \subseteq S(p,b)$.

For $\eta \in \mathbb{Q}_+$ define the (finite) rational grid

$$G_\eta := \{(\eta I)e + p : e \in \mathbb{Z}^2\} \cap S(p,b),$$

where I denotes the identity matrix.

Given $\eta \in \mathbb{Q}_+$, we can run the following algorithm:

> `Dovetail the computation of` ω_A `on all vertices of` G_η `until for every vertex` q`, for which all neighbours belong to` G_η`,` ω_A `is defined on` q `or on one of` q`'s neighbours.`

Termination of the algorithm is implied by the fact that if ω_A is undefined on all neighbours of a point q, all these neighbours lie on the boundary of A, and hence q itself must be an interior point of A , for which computation of ω_A terminates.

Let T be the set of all vertices $q \in G_\eta$ for which we discovered that $\omega_A(q) = 1$. Then $conv(T)$ is (by definition) a subset of A. How well does $conv(T)$ approximate A?

Define $C(x) := conv\{S(p,a), x\}$. Remember that $S(p,a)$ is contained in A. Hence, if $x \in A$, then $int(C(x)) \subseteq int(A)$. An elementary geometric argument verifies the following statement.

Claim There is a constant $c > 0$ such that for all $\eta > 0$ and all $x \in A$ there is a grid point $q \in C(x)$ which belongs to T and has distance less than $c\eta$ from x.

The reason for this is that if η is small there are enough grid points in $C(x)$ close to x. To provide for large η we choose c such that $q := p$ will do. The claim immediately implies that $d_H(A, conv(T)) < c\eta$.

So by choosing η small enough we can approximate A from the inside to any given accuracy.

For approximations from the outside consider the set

$$T' := \{q \in G_\eta : d(q, conv(T) \leq (c+2)\eta\}$$

of all grid points having distance less than $(c+2)\eta$ from $conv(T)$. We claim that

(1) $A \subseteq conv(T')$, and

(2) $d_H(conv(T'), A) \leq (c+2)\eta$.

Since T' is effectively computable in η, $conv(T')$ is the desired approximation from the outside.

To prove (1) let x be a point of A. Then $d(x, conv(T)) \leq d_H(A, conv(T)) \leq c\eta$ and hence there is a point $y \in conv(T)$ with $d(x, y) < c\eta$. The point y has four neighbours $\{q_1, \ldots, q_4\}$ in the grid G_η, so $y \in conv\{q_1, \ldots, q_4\}$ and the distance of y to each of the points $q_1, \ldots, q_4$ is less than $\sqrt{2}\eta < 2\eta$. This implies that the distance of x to each of these four points is less than $c\eta + 2\eta = (c+2)\eta$. Thus all of $q_1, \ldots, q_4$ belong to T' and therefore x lies in $conv(T')$ proving (1). Now (2) is immediate using Fact 2.1.

$(ii) \Rightarrow (iii')$ The following algorithm will compute the WMT α_A of (iii') given effective rational approximations of A:

```
Search for an i such that S(x,r+2^-i) ⊆ P_i or S(x,r) ∩ Q_i = ∅.
In the first case set α(x,r) to 1, else define it to be 0.
```

By (ii) the polygons P_i and Q_i are computable effectively in i, so the description above is indeed an algorithm and one easily checks that it is correct.

We are left with the proof that the search terminates if and only if $(x, r) \in dom(\alpha_A)$. The "only if" part is obvious. So let α_A be defined on (x, r). Consider the case that $\alpha_A(x, r) = 0$, that is $S(x, r) \subseteq ex(A)$. Then $d(x, A) > r$ since $S(x, r)$ is compact. Let $\varepsilon := d(x, A) - r > 0$. We claim that as soon as $d_H(P_i, Q_i) < \varepsilon$ the condition $S(x, r) \cap Q_i = \emptyset$ is met, because a point y in $S(x, r) \cap Q_i$ would have distance $d(y, A) \geq d(x, A) - d(x, y) > \varepsilon$ from A which contradicts $d(y, A) \leq d_H(Q_i, A) \leq d_H(Q_i, P_i) < \varepsilon$.

Finally suppose that $\alpha_A(x, r) = 1$, that is $S(x, r) \subseteq int(A)$. In this case $d(x, \partial A) > r$, so $\varepsilon := d(x, \partial A) - r > 0$. Let $d_H(P_i, Q_i) < \varepsilon - 2^{-i}$ from which $d_H(A, P_i) < \varepsilon - 2^{-i}$ is immediate. For an arbitrary point $z \in S(x, r + 2^{-i})$ the following holds:

$$\begin{aligned} d(z, \partial A) &\geq d(x, \partial A) - d(x, z) = (r + \varepsilon) - (r + 2^{-i}) \\ &= \varepsilon - 2^{-i}. \end{aligned}$$

Thus $z \in A_{-(\varepsilon - 2^{-i})}$ implying that $z \in P_i$ by Lemma 2.5, thereby proving $S(x, r + 2^{-i}) \subseteq P_i$.

$(ii) \Rightarrow (iv')$ For LITs we use an argument similar to that in $(ii) \Rightarrow (iii')$. The algorithm to compute $\gamma_A(x,y)$ is:

> `Search for an` i `such that either` $l(x,y) \cap int(P_i) \neq \emptyset$ `or` $l(x,y) \cap Q_i = \emptyset$`. In the first case set` $\gamma(x,y)$ `to` 1`, else define it to be` 0`.`

Proofs of correctness and termination are analogous to the ones above (Lemma 2.5 is not needed here).

$(iii') \Rightarrow (iii)$ **and** $(iv') \Rightarrow (iv)$ Trivial.

$(iii) \Rightarrow (i)$ Let A be a convex region which contains the disc $S(p,a)$ in its interior. For a point $q \in \mathbb{Q}^2$ define the two sets

$$
\begin{aligned}
C(q) &:= conv\{S(p,a), q\} \\
D(q) &:= \{z \in \mathbb{R}^2 : (\exists y \in S(p,a))[q \in yz]\}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
q \in A &\Rightarrow int(C(q)) \subseteq int(A) \\
q \in cl(ex(A)) &\Rightarrow int(D(q)) \subseteq ex(A)
\end{aligned}
$$

This gives us a way to compute $\omega_A(q)$ given a partial recursive WMT α_A:

> `On input` $q \in \mathbb{Q}^2$ `search for` $(x,r) \in \mathbb{Q}^2 \times \mathbb{Q}_+$ `with either`
>
> `(1)` $S(x,r)$ `lies in the interior of` $C(q)$ `and` $\alpha_A(x,r) = 0$`, or`
>
> `(2)` $S(x,r)$ `lies in the interior of` $D(q)$ `and` $\alpha_A(x,r) = 1$`,`
>
> `by dovetailing the computation of` α_A`. In case` (1) `let` $\omega(q)$ `be` 0`, otherwise let` $\omega(q)$ `be` 1`.`

We claim that ω is the weak characteristic function of A. First of all assume the algorithm terminates by finding (x,r) as in (1). For contradiction suppose $q \notin ex(A)$ that is $q \in A$. But then $int(C(q))$ lies in $int(A)$ and therefore $S(x,r)$ does, which is at odds with $\alpha_A(x,r) = 0$. Case (2) is similar. Now suppose the algorithm does not terminate. In this case any disc contained in the interior of $C(q)$ contains an interior point of A and similarly any disc contained in the interior of $D(q)$ contains an exterior point of A. Since these discs can be chosen arbitrarily close to q this means that q is a boundary point (having interior and exterior points in every neighbourhood). This finishes the proof.

$(iv) \Rightarrow (ii')$ Given that we can decide whether a line intersects the set A it seems natural to approximate A by cutting off small parts and thereby localizing boundary points. Lemma 2.13 identified boundary points which can be "cut off" as extreme points.

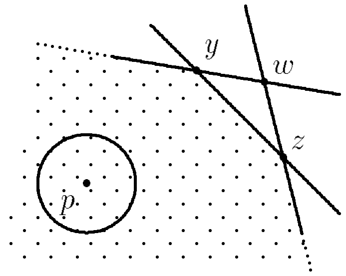

Figure 2: Localizing extreme points

We also know that finite sets of extreme points are sufficient to approximate A to any given accuracy (Theorem 2.9).

So our first goal will be to give an effective method to localize extreme points. In a second step we use these localizations to approximate A.

Let A be a convex region with a LIT γ_A that is partial recursive. Choose a rational point p and $a, b \in \mathbb{Q}_+$ such that

$$S(p,a) \subseteq A \subseteq S(p,b-1) \ .$$

The main work of localizing extreme points is done by triples (w,y,z) of rational points which satisfy

(1) $\gamma_A(y,w) = 0$ ($\Rightarrow l(y,w) \cap int(A) = \emptyset$)

(2) $\gamma_A(z,w) = 0$ ($\Rightarrow l(z,w) \cap int(A) = \emptyset$)

(3) $\gamma_A(y,z) = 1$ ($\Rightarrow l(y,z) \not\subseteq ex(A)$)

(4) pw intersects yz.

Figure 2 shows the relative position of the points.

Fact 2.12 implies that the triangle with vertices w, y and z contains an extreme point of A. So $S(w, r(w,y,z))$ - setting $r(w,y,z) := 2\max\{d(w,y), d(w,z)\}$ - even contains an extreme point of A *in its interior.*

The following lemma justifies our hope that we can effectively localize extreme points in this manner.

Lemma 5.2 *For every extreme point and any $\delta > 0$ there are rational points w, y and z with distance less than δ from x which fulfill* (1) - (4).

Proof The core of this proof is the "only if" direction of Lemma 2.13 which tells us that for any extreme point x and any $\eta > 0$ there is a rational halfplane H containing x for which $H \cap A \subseteq S(x, \eta/2)$. Let η be less than $a/2$. Let h be the line bounding H. Then h is rational and rational points are dense on h. Since

$h \cap A \subseteq S(x, \eta/2)$ there are rational points y and z on h with distance less than η from x which do not lie in A. Since $x \in H$, $p \notin H$ (because η is less than $a/2$) and $xp \subseteq A$), the line h contains a point of A which must lie on yz. By varying y and z slightly (without changing $y, z \notin A$ and $d(y, x) < \eta$, $d(z, x) < \eta$) we can achieve that yz even contains an interior point of A. Choose lines through y and z which do not intersect A (since $y, z \notin A$ and A is compact). These lines intersect in a point w lying outside of A. The radius $r(w, y, z)$ can be made arbitrarily small by choosing η small enough. This proves the lemma. ∎

The results in Section 2.3 together with the last lemma suggest a certain approach: search for approximations of extreme points and stop when their convex hull is close enough to A.

This approach poses two problems: by Theorem 2.9 we know that a good polygonal approximation of A exists which has only extreme points as vertices. But will the approximation remain good enough if extreme points are substituted by the results from the localization construction? And secondly: how do we know that an approximation we have found is good enough? The first question is answered by Fact 2.2. To test the quality of the approximation we will use approximations of A from the outside: if a rational polygon Q contains A in its interior we can easily check this: test whether an arbitrary (but fixed) point of A lies in Q and no side of Q intersects A (do this via the LIT γ_A). So there is an effective enumeration $(Q_k)_{k \in \mathbb{N}}$ of all rational polygons containing A.

We can now write down an algorithm to find a polygonal approximation with accuracy 2^{-i}; to achieve this we compute a finite set W of approximations of extreme points such that $conv(W)$ and A are close.

> $V := \emptyset$;
> `Dovetail the computation of` γ_A `to find triples` (w, y, z)
> `satisfying conditions` (1) `-` (4) `and` $r(w, y, z) \leq 2^{-(i+4)}$.
>   `For each such triple found update`
> $V := V \cup \{w\}$.
> `Simultaneously search for` $W \subseteq V$ `and` $k \in \mathbb{N}$
> `such that` $d_H(conv(W), Q_k) \leq 2^{-(i+1)}$.
>   `If the search terminates output` Q_k.

Assume that the algorithm terminates with output k such that $d_H(conv(W), Q_k) \leq 2^{-(i+1)}$, then it is correct:

We know that for each $w \in W$ there is an $x \in A$ for which $d(w, x) < r(w, y, z) \leq 2^{-(i+4)}$. Let X be the set containing these x's. Fact 2.2 implies

$$d_H(conv(W), conv(X)) \leq d_H(W, X) \leq 2^{-(i+4)}.$$

Note that by definition $X \subseteq A$ and thus also $conv(X) \subseteq A$ which in turn is a subset of Q_k. Using Fact 2.3 we obtain

$$\begin{aligned} d_H(conv(W), A) &\leq \max\{d_H(conv(W), conv(X)), d_H(conv(W), Q_k)\} \\ &= 2^{-(i+1)} < 2^{-i}. \end{aligned}$$

We will now prove that the algorithm terminates. By Theorem 2.9 there is a finite set $D := \{x_1, \ldots, x_n\}$ of extreme points such that

$$d_H(conv(D), A) < 2^{-(i+3)}.$$

Because of Lemma 5.2 for every x_i there is a triple (w_i, y_i, z_i) such that $x_i \in S(w_i, r(w_i, y_i, z_i))$, $r(w_i, y_i, z_i) \leq 2^{-(i+4)}$ and the triple fulfills $(1) - (4)$.

Since the algorithm checks all triples there is a point in the computation after which V contains $W := \{w_1, \ldots, w_n\}$. Note that

$$d_H(conv(D), conv(W)) \leq d_H(D, W) \leq 2^{-(i+4)}.$$

The proof of $(i) \Rightarrow (ii)$ showed that there are infinitely many rational polygons Q for which $d_H(A, Q) < 2^{-(i+2)}$ and no side of Q intersects A. So there is a k for which W is checked against such a Q_k. Since under these circumstances

$$\begin{aligned} d_H(conv(W), Q_k) &\leq d_H(conv(W), conv(D)) + d_H(conv(D), A) + d_H(A, Q_k) \\ &\leq 2^{-(i+4)} + 2^{-(i+3)} + 2^{-(i+2)} < 2^{-(i+1)} \end{aligned}$$

the algorithm stops and the proof of $(iv) \Rightarrow (ii')$ is finished.

$(ii') \Rightarrow (vi)$ Define an Oracle Turing Machine M^Φ, which on input n computes $d(\Phi(n+2), P_{n+2})$ with accuracy $2^{-(n+1)}$ and prints out the result. This is possible, since $\Phi(n+2)$ and P_{n+2} are rational.

Assume that Φ is a representation of x. An easy computation shows that $|d(x, P_{n+2}) - d(x, A)| < 2^{-(n+2)}$ (since $d_H(A, P_{n+2}) < 2^{-(n+2)}$), so

$$\begin{aligned} |M^\Phi(n) - d(x, A)| &\leq |M^\Phi(n) - d(\Phi(n+2), P_{n+2})| \\ &\quad + |d(\Phi(n+2), P_{n+2}) - d(x, P_{n+2})| \\ &\quad + |d(x, P_{n+2}) - d(x, A)| \\ &\leq 2^{-(n+1)} + d(\Phi(n+2), x) + 2^{-(n+2)} \\ &\leq 2^{-n}, \end{aligned}$$

which proves that the algorithm is correct.

$(vi) \Rightarrow (iii)$ To prove this implication it must be shown that a WMT is computable if A is Turing located. For a pair $(x, r) \in \mathbb{Q}^2 \times \mathbb{Q}_+$ compute $d \in \mathbb{Q}_+$ such that $|d(x, A) - d| < \frac{r}{3}$ (this is effectively possible since A is Turing located). Define $\alpha_A(x, r)$ to be 1 if $d < \frac{r}{2}$ and 0 else. We have to check that this is indeed a WMT:

- $S(x, r) \subseteq int(A) \Rightarrow d(x, A) = 0 \Rightarrow d < \frac{r}{3} < \frac{r}{2} \Rightarrow \alpha_A(x, r) = 1$
- $S(x, r) \subseteq ex(A) \Rightarrow d(x, A) > r \Rightarrow d > \frac{2}{3}r > \frac{r}{2} \Rightarrow \alpha_A(x, r) = 0$

$(iii) \Leftrightarrow (v)$ Let the WMEM problem be solvable by β_A. We claim that β_A is a WMT:

- $S(x, r) \subseteq int(A) \Rightarrow \beta_A(x, r) \neq 0 \Rightarrow \beta_A(x, r) = 1$
- $S(x, r) \subseteq ex(A) \Rightarrow \beta_A(x, r) \neq 1 \Rightarrow \beta_A(x, r) = 0$

On the other hand the existence of a partial recursive WMT implies (ii). So we can effectively approximate A by P_i and Q_i from inside and outside. To solve the Weak Membership Problem we wait until $S(x,r)$ intersects either P_i or Q_i^c. In the first case we let β_A be 1 and 0 else. The computation terminates, because $d_H(P_i,Q_i) \to 0$ (for $x \in A$ use Lemma 2.5). It is easy to see that the resulting function solves WMEM.

This finishes the proof. ▮

Remarks

(1) The proof of the theorem showed that (ii) can be substituted by a seemingly weaker variant:

(ii') There is a computable sequence $(P_i)_{i\in\mathbb{N}}$ of rational polygons such that $d_H(P_i,A) < 2^{-i}$ for all i.

We actually proved $(ii) \Rightarrow (ii') \Rightarrow (vi)$ to integrate (ii') into the scheme. The direct implication $(vi) \Rightarrow (ii)$ was shown in [3]. Furthermore with the help of Lemma 2.5 and techniques established in the proof of $(i) \Rightarrow (ii)$ a direct proof of $(ii') \Rightarrow (ii)$ would not have been difficult: from an approximation in the sense of (ii') inner and outer approximations are constructible.

(2) We have the following, by no means obvious, result:

A convex region is Turing located iff its complement is Turing located.

If the complement of a set is Turing located we can show as in the proof of $(vi) \Rightarrow (iii)$ that a WMT is computable. On the other hand we have $(vi) \Rightarrow (ii)$ and (ii) implies that the complement of a convex region is Turing located as in $(ii) \Rightarrow (vi)$.

(3) Grötschel, Lovász and Schrijver in their book [4] take WMEM as a starting point to prove further equivalent variants (even preserving polynomial time computability) like optimization of linear functions, separation by hyperplanes and others.

6 Approximating Extreme Points

In the proof of $(iv) \Rightarrow (ii')$ we came very close to establishing an effective version of the Minkowski theorem. Actually such an effective version of the theorem was proved by Ge and Nerode in [3] starting from the concept of "Turing located". We give a new proof of this result which is based on techniques used in the theorem.

Corollary 6.1 [3] *If A is a computable convex region then there is a uniformly computable sequence of real points $\{y_i\}_{i\in I}$ such that their closure is the set of extreme points of A and there is a computable function $g(i)$ such that $d_H(conv\{y_1,\ldots,y_{g(i)}\},A) \le 2^{-i}$. If $ext(A)$ is of cardinality n then so is I. Otherwise I is $\mathbb{N}$.*

The idea of the proof is to make use of the construction described in part (b) of $(iv) \Rightarrow (ii')$. The main difference in applying this construction is that we have to compute convergent approximations of single extreme points simultaneously for a dense subset of $ext(A)$. So the computation of an approximation with accuracy 2^{-i} will depend on all previous computations with lower accuracy. We therefore choose the natural approach of improving approximations stepwise.

Lemma 6.2 *Suppose A is a convex region, for which a LIT is partial recursive, and we are given $(S(w_k, r_k))_{k=1}^{n}$ such that*

- *all $S(w_k, r_k)$ contain an extreme point in their interior, are pairwise disjoint, and*
- $r_k \leq 2^{-i}$.

Then we can effectively find $(S(w'_k, r'_k))_{k=1}^{n'}$ with $n' \geq n$, and

- *$S(w'_k, r'_k) \subseteq S(w_k, r_k)$ for all k with $1 \leq k \leq n$,*
- *all $S(w'_k, r'_k)$ contain an extreme point in their interior, are pairwise disjoint, and*
- *$r'_k \leq 2^{-(i+1)}$, and*
- $d_H(conv\{w'_1, \ldots, w'_{n'}\}, A) \leq 2^{-(i+1)}$.

This lemma will be the main tool in the proof of the corollary.

Proof of Corollary 6.1 Since we assumed A to be computable, a LIT is partial recursive by Theorem 5.1. We can therefore apply the lemma as follows: initially start the algorithm with the empty sequence $(S(w_k^{(1)}, r_k^{(1)}))_{k=1}^{n^{(1)}}$ putting $n^{(1)} = 0$ and repeat it for every $i \in \mathbb{N}$. Set $S(w_k^{(i+1)}, r_k^{(i+1)}) := S(w'_k, r'_k)$ for every k with $1 \leq k \leq n(i+1) := n'$.

If $n(i)$ is unbounded put $I := \mathbb{N}$ otherwise let I be $\{1, \ldots, \sup\{n(i), i \in \mathbb{N}\}\}$. Define

$$\{y_k\} := \bigcap_{i:k \leq n(i)} S(w_k^{(i)}, r_k^{(i)}) \text{ for all } k \in I.$$

The point y_k is well-defined since $S(w_k^{(i)}, r_k^{(i)})$ is a descending sequence of sets with diameters converging to zero. The lemma implies that $(y_k)_{k \in I}$ is a uniformly computable sequence of real points.

Putting $g(i) := n(i+1)$ which is clearly computable we have

$$\begin{aligned} d_H(conv\{y_1, \ldots, y_{g(i)}\}, A) &\leq d_H(conv\{y_1, \ldots, y_{g(i)}\}, conv\{w_1^{(i+1)}, \ldots, w_{n(i+1)}^{(i+1)}\}) \\ &\quad + d_H(conv\{w_1^{(i+1)}, \ldots, w_{n(i+1)}^{(i+1)}\}), A) \\ &\leq 2^{-(i+1)} + 2^{-(i+1)} = 2^{-i}. \end{aligned}$$

By Fact 2.7 the set $ext(A)$ is closed. Since every $S(w_k^{(i)}, r_k^{(i)})$ contains an extreme point this means that y_k belongs to $ext(A)$.

We will now prove that $(y_k)_{k \in I}$ is even a dense subset of $ext(A)$. Let x be an arbitrary point of $ext(A)$. We have to show that every $S(x, \varepsilon)$ contains a y_k. Lemma 2.13 tells us that there is a halfplane H containing x such that $H \cap A \subseteq S(x, \varepsilon)$. We may assume that x lies in the interior of H. If there were no y_k in $S(x, \varepsilon)$ then all y_k must lie in H^c. This would imply that $d_H(conv\{y_1, \ldots, y_i\}, A) \geq d(x, H^c) =: \delta > 0$ for all i, which contradicts the inequality we just proved. ∎

We are left with the proof of the lemma.

Proof of Lemma 6.2 We write down an algorithm solving the problem. As in the proof of $(iv) \Rightarrow (ii')$ we let $(Q_j)_{j\in\mathbb{N}}$ be an effective enumeration of all polygons containing A. Conditions (1) - (4) mentioned in the algorithm are those defined in the proof of $(iv) \Rightarrow (ii')$.

The algorithm has two parts. The first part handles the $(S(w_k, r_k))_{k=1}^n$ which are given as input, and the second part searches for an approximation incorporating the results of part one.

```
1st Part   Let all k with 1 ≤ k ≤ n be unmarked.
           Dovetail the computation of γ_A to find triples (w,y,z)
           satisfying conditions (1) - (4) and r(w,y,z) ≤ 2^-(i+4).
                If S(w,r(w,y,z)) is contained in any S(w_k,r_k) with
                unmarked k, then set
                   S(w'_k,r'_k) := S(w,r(w,y,z)).
           Stop dovetailing if all k are marked.

2nd Part   Dovetail the computation of γ_A to find a set
           W = {(w'_{n+1},y'_{n+1},z'_{n+1}),...,(w'_{n'},y'_{n'},z'_{n'})} such that
           all (w,y,z) ∈ W satisfy (1) - (4) and r(w,y,z) ≤ 2^-(i+4) and
           all S(w'_k,r(w'_k,y'_k,z'_k)) are pairwise disjoint (1 ≤ k ≤ n').
           Simultaneously search for some j ∈ ℕ such that
                d_H(conv{w'_1,...,w'_{n'}},Q_j) ≤ 2^-(i+4)
           and terminate with output j if the search is successful.
```

The first part of the algorithm terminates since by assumption every $S(w_k, r_k)$ contains an extreme point in its interior, so by Lemma 5.2 there is a triple (w, y, z) fulfilling all conditions.

If the second part of the algorithm terminates, the whole algorithm is correct. For this we use the same argument as in $(iv) \Rightarrow (ii')$: for every w'_k $(1 \le k \le n')$ there is an $x_k \in A$ with $d(w'_k, x_k) < 2^{-(i+4)}$. Set $X := \{x_k : 1 \le k \le n'\}$. By Fact 2.2 we have

$$\begin{aligned} d_H(conv\{w'_1, \ldots, w'_{n'}\}, conv(X)) &\le d_H(\{w'_1, \ldots, w'_{n'}\}, X) \\ &\le 2^{-(i+4)} \end{aligned}$$

Moreover, by the condition of termination, $d_H(conv\{w'_1, \ldots, w'_{n'}\}, Q_j) \le 2^{-(i+4)}$ for some j. As X and therefore $conv(X)$ are subsets of A, Fact 2.3 implies

$$\begin{aligned} d_H(conv\{w'_1, \ldots, w'_{n'}\}, A) &\le \max\{d_H(\{w'_1, \ldots, w'_{n'}\}, conv(X)), \\ &\qquad d_H(\{w'_1, \ldots, w'_{n'}\}, Q_j)\} \\ &\le 2^{-(i+1)} \end{aligned}$$

We are left with the proof that the second part of the algorithm terminates. By Theorem 2.9 we know that there is a finite subset D of $ext(A)$ such that $d_H(conv(D), A) \le 2^{-(i+2)}$.

We partition D into two subsets D_1 and D_2 as follows:

$$\begin{aligned} D_1 &:= \{x \in D : (\exists k)[1 \le k \le n \wedge x \in S(w'_k, r'_k)]\} \\ D_2 &:= D - D_1 \end{aligned}$$

D_1 contains those points of D which are already taken care of. Now for every point x in D_2 there are infinitely many triples (w, y, z) such that x lies in the interior of $S(w, r(w, y, z))$ and $r(w, y, z) \leq 2^{-(i+4)}$. Thus there are infinitely many sets $W = \{(w'_{n+1}, y'_{n+1}, z'_{n+1}), \ldots, (w'_{n'}, y'_{n'}, z'_{n'})\}$ (where $n' = |D_2| + n$) with

$$d_H(W, D_2) \leq 2^{-(i+4)}.$$

For every such W we have

$$\begin{aligned} d_H(\{w'_1, \ldots, w'_n\} \cup W, D_1 \cup D_2) &\leq \max\{d_H(\{w'_1, \ldots, w'_n\}, D_1), d_H(W, D_2)\} \\ &\leq 2^{-(i+4)}, \end{aligned}$$

and therefore by Fact 2.2

$$d_H(conv(\{w'_1, \ldots, w'_n\} \cup W), conv(D_1 \cup D_2)) \leq 2^{-(i+4)} \tag{1}$$

Furthermore there are infinitely man Q_j with

$$d_H(Q_j, A) \leq 2^{-(i+3)} \tag{2}$$

Since every W is tested against all previous Q_j the algorithm must reach a stage, when a W fulfilling (1) is tested against a Q_j fulfilling (2). In this case

$$\begin{aligned} d_H(conv(\{w'_1, \ldots, w'_n\} \cup W), Q_j) &\leq d_H(conv(\{w'_1, \ldots, w'_n\} \cup W), conv(D)) \\ &\quad + d_H(conv(D), A) + d_H(A, Q_j) \\ &\leq 2^{-(i+4)} + 2^{-(i+2)} + 2^{-(i+3)} \\ &< 2^{-(i+1)} \end{aligned}$$

and the algorithm terminates. This completes the proof of the Lemma. ∎

7 Recursively Enumerable Convex Sets

At this point the question of recursive enumerability of convex sets suggests itself. Adhering to our concept (excluding the boundary) we call a set $A \subseteq \mathbb{R}^2$ *recursively enumerable* if the rational interior points of A are recursively enumerable. We have a list of results rather similar to the ones above.

Theorem 7.1 *Let A be a convex region. Then the following are equivalent:*

(i) A is recursively enumerable.

(ii) There is a computable sequences $(P_i)_{i \in \mathbb{N}}$ of rational polygons such that $P_i \subseteq A$ and $d_H(P_i, A) \to 0$ as $i \to \infty$.

(iii) There is an effective enumeration of all $(x, r) \in \mathbb{Q}^2$ for which $S(x, r) \subseteq int(A)$.

(iv) There is an effective enumeration of all $(x, y) \in \mathbb{Q}^2$ for which $l(x, y) \cap int(A) \neq \emptyset$.

(v) *The function $x \mapsto d(x, A)$ is weakly computable, that is there is an OTM M such that $|M^{\Phi}(n) - d(x, A)|$ converges to zero as $n \to \infty$, where Φ is any oracle representing x.*

Proof All implications except $(iv) \Rightarrow (i)$ are straightforward. The construction used in Theorem 5.1, $(iv) \Rightarrow (ii')$ will not work here as it made use of lines *not* intersecting A. Thus the need for a new proof arises:

$(iv) \Rightarrow (i)$ Let $S(p, a) \subseteq A \subseteq \triangle$ where p and a are rational and $\triangle$ is a rational triangle. Let $q \neq q'$ be two rational points such that q lies on pq' and consider the following algorithm.

```
xy := line segment through q' which is perpendicular to pq'
      and x,y lie on the boundary of △
if γ_A(x,y) = 1 and test(x) and test(y) then
   return ``q ∈ int(A)''
else
   diverge.

procedure test(z)
   set y :=intersection of the ray from z through q
           with the boundary of △;
   set x :=intersection of the ray from y through q'
           with the boundary of △;
   if xy intersects S(p,a) then
      return true;
   if γ_A(x,y) = 1 and test(x) then
      return true
   else
      return false;
end test.
```

Figure 3 illustrates a run of $test(x)$.

We claim first that for every $q \in int(A)$ there is a q' as above such that the algorithm terminates with the message $q \in int(A)$ and secondly that termination with message $q \in int(A)$ implies that q is indeed an interior point of A. This is termination and correctness respectively.

We prove correctness first. The underlying idea is that for every line segment xy through q' with $\gamma_A(x, y) = 1$ either $q'x$ or yq' must contain a point of A. By induction we prove that after $test(x)$ has returned "true", either yq' contains a point of A or $q \in int(A)$. We write x' and y' for the x and y computed in the procedure $test$ and simply x and y for the variables of the calling procedure (be it the main program or another call of $test$). The recursive calls only stop, when $x'y'$ intersects $S(p, a)$. This means that (by construction) $y'q'$ intersects $S(p, a)$ and therefore contains a point of A. Then there are two possibilities: either $q'x$ contains a point of A or it does not. In the first case we have $q \in int(A)$ because we have seen that there is a point of A on either side of $l(p, q)$ and both lie on the far side of $l(x, y')$ from p, so

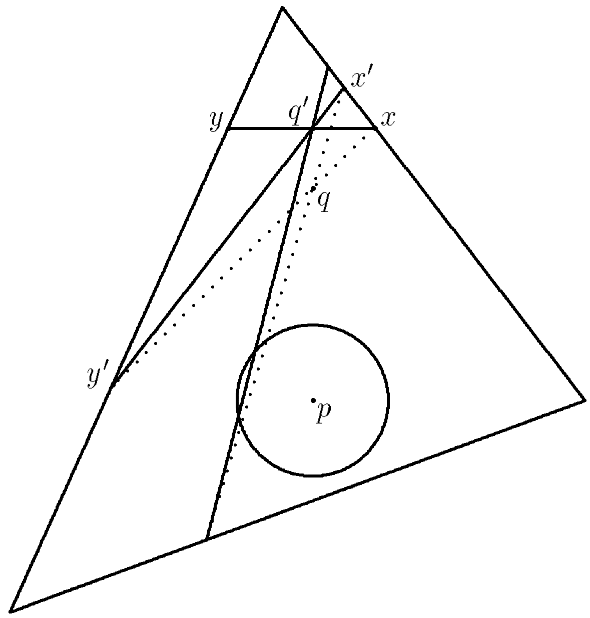

Figure 3: Testing whether $q \in int(A)$

the convex hull of both points with p must contain q, which therefore lies in $int(A)$. If on the other hand $q'x$ does not contain a point of A then yq' must intersect A (since xy does). The proof of the induction step is identical.

So start the algorithm with xy and suppose that $\gamma_A(x, y) = 1$ and $test(x)$ and $test(y)$ are true. By the statement proved above we have that either $q \in int(A)$ or ($q'x$ intersects A and yq' intersects A). The second statement implies $q \in int(A)$ as above.

To prove termination suppose that $q \in int(A)$. Then there is a rational point $q' \in int(A)$ different from q with $q \in pq'$. But as all lines that are tested by the algorithm run through q' every computation of γ_A terminates with value 1. So the only way for the algorithm to diverge is that test is called infinitely often, that is ever new line segments are tested which never intersect $S(p, a)$. If this were the case the test lines would converge to a limit line which has two characteristics: it is a fixed point of the construction and it does not intersect $S(p, a)$. But this is impossible since the only fixed point of the construction is the line through $q'q$ which intersects $S(p, a)$ because it runs through p.

This finishes the proof.

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Remark: The previous construction can be used to give an alternative proof of Theorem 5.1, $(iv) \Rightarrow (ii')$. However, this proof would yield no information of how to approximate extreme points.

8 Conclusion

We have presented a natural notion of computable convex set that is invariant under various alternative definitions.

One may well ask what would happen, if we used one of the other models customary in computable geometry such as Recursive Analysis or the BSS-model [1]. There are two questions: (1) Does Theorem 5.1 translate (with necessary adjustments)? and (2) How do the notions of computable convex sets in different models relate? To give a short answer: recursive analysis adds nothing new, the theorem translates with minor changes and the convex sets that are computable are the same as in our rational model. Things are different in the BSS-model. Again the theorem translates, with the exception of item (vi) which is firmly rooted in recursive analysis. Convex sets computable in the rational model remain computable in the BSS-model, but (not surprisingly, because this model allows programs to use arbitrary real constants) more convex sets are computable (for example the disc $S(0, r)$ with non-computable real radius r).

However, we believe that the rational model of computation is a very appropriate one for digital computers. It would be interesting to investigate the effective content of convex geometry using this model. The current paper provides a starting point.

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