Notre Dame Journal of Formal Logic

Cuppability of Simple and Hypersimple Sets

Martin Kummer and Marcus Schaefer

Abstract An incomplete degree is *cuppable* if it can be joined by an incomplete degree to a complete degree. For sets fulfilling some type of simplicity property one can now ask whether these sets are cuppable with respect to a certain type of reducibilities. Several such results are known. In this paper we settle all the remaining cases for the standard notions of simplicity and all the main strong reducibilities.

There are two sides to every question.
—Protagoras, quoted in Diogenes Laertius, *Lives of Eminent Philosophers.*

1 Introduction

In his approach to constructing an incomplete c.e. degree, Emil Post attempted to define structural properties of c.e. sets that would force their incompleteness. In his groundbreaking 1944 paper *Recursively enumerable sets of positive integers and their decision problems* ([24], reprinted in Davis's *The Undecidable* [1]) this goal led him to isolate many of the classical concepts of computability, including creativity, many-one reducibility, bounded and unbounded truth-table reducibility, simplicity, hypersimplicity, and hyperhypersimplicity.

Among other results, Post showed that simple sets exist and cannot be bounded truth-table complete. In other words, the *btt*-degree $\mathbf{a}$ of a simple set is an *intermediate* *btt*-degree: $\mathbf{0}_{btt} <_{\mathrm{btt}} \mathbf{a} <_{\mathrm{btt}} \mathbf{0'}_{btt}$. More is true: the *btt*-degree of a simple set cannot even be joined with another bounded truth-table degree below $\mathbf{0'}_{btt}$ to yield the complete degree: if $\mathbf{a}$ is a simple *btt*-degree and $\mathbf{a} \cup \mathbf{b}$ is the complete *btt*-degree, then $\mathbf{b}$ is the complete *btt*-degree. We say that simple sets are not *btt-cuppable* [7; 8; 10; 26].

Printed August 15, 2006
2001 Mathematics Subject Classification: Primary, 03D30; Secondary, 03D25, 03D28
Keywords: Cuppability, completeness, strong reducibilities, simple sets, hypersimple sets

Post also showed that hypersimple sets exist and cannot be truth-table complete. In degree-theoretic terms, hypersimple degrees are of intermediate truth-table degree. But are they *tt*-cuppable? It is known that hypersimple degrees are not *wtt*-cuppable [9], but *tt*-cuppability has been a long-standing open question.

In this paper we investigate how structural notions such as simplicity and hypersimplicity force the non-cuppability of degrees for different types of reductions, generalizing the classical incompleteness results.

Definition 1.1 A set A is called *r-cuppable in the c.e. degrees*, or *cuppable in the c.e. r-degrees*, if there is a c.e. set B such that $K \leq_{\mathrm{r}} A \oplus B$ and $K \not\leq_{\mathrm{r}} B$, where r is a class of reductions such as many-one (m), bounded truth-table (btt), bounded disjunctive (bd), conjunctive (c), disjunctive (d), positive (p), parity ($\oplus$), truth-table (tt), weak truth-table (wtt), Q, Turing (T) reductions.

We call A *r-cuppable*, if there is a set B such that $K \leq_{\mathrm{r}} A \oplus B$ and $K \not\leq_{\mathrm{r}} B$, where r is a class of reductions.

We define the less familiar reductions in Section 2.

The following proposition summarizes the known results on the cuppability of degrees.

Proposition 1.2

- *A simple set is not m-cuppable* (Lachlan [14]).
- *A simple set is not btt-cuppable* (Downey [7], Schaefer [26], also see Theorem 3.8).
- *A hypersimple set is not c-cuppable* (Degtev [3]).
- *A hypersimple set is not wtt-cuppable in the c.e. degrees* (Downey, Jockusch [9]).
- *A K-hypersimple set is not e-cuppable to $\overline{K}$ in the Σ^0_2 enumeration degrees* (Nies, Sorbi [20]).

The most notable omission in this series of results is the case of truth-table reductions. We will settle the general truth-table case in Section 6: in contrast with the result that hypersimple sets cannot be *wtt*-cuppable, they *can* be *tt*-cuppable, even in the c.e. degrees. However, dense simple sets are not *tt*-cuppable (Theorem 6.6).

We discuss special truth-table reducibilities in Section 3, which covers conjunctive, disjunctive, and bounded truth-table reductions, Section 4 on positive reductions, and Section 5 on the parity reduction. Q-reducibility is investigated in Section 7; for Q-reducibility the non-cuppability turns into an even stronger non-splitting theorem: if $\mathbf{a} = \mathbf{b} \cup \mathbf{c}$ is Q-complete, then either $\mathbf{b}$ or $\mathbf{c}$ is Q-complete.

In Section 7 we also show that Marchenkov's result that a c.e., semirecursive, η-maximal set is not Turing-complete does not yield a non-cuppability result for c.e. T-degrees; it does, however, give non-cuppability results for p-, tt-, and wtt-degrees as we show in Section 6.3.

A detailed account of Post's program can be found in [21, Chapter 3].

2 Terminology and Notation for Reductions

We write $(D_i)_{i\in\omega}$ for the *canonical enumeration of the finite sets* and $(W_i)_{i\in\omega}$ for the *standard enumeration of the c.e. sets.*

We will use χ_A to be the *characteristic function of* A, that is,

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{else,} \end{cases}$$

as well as the *characteristic vector of* A, that is

$$\chi_A(x_1, \ldots, x_n) = (\chi_A(x_1), \ldots, \chi_A(x_n)).$$

For any finite set $D = \{x_1 < \ldots < x_n\}$, let $\chi_A(D) = \chi_A(x_1, \ldots, x_n)$.

In the context of numbers, the symbol $\oplus$ will denote the parity operator, that is, addition modulo 2.[1]

(bounded truth-table): $A \leq_{\mathrm{btt}} B$ if there is a computable f and a computable $\alpha : \omega \times \{0,1\}^k \to \{0,1\}$ (for some k) so that $|D_{f(x)}| = k$ and

$$x \in A \iff \alpha_x(\chi_B(D_{f(x)})) = 1,$$

writing $\alpha_x(v)$ for $\alpha(x, v)$.

(bounded disjunctive): $A \leq_{\mathrm{bd}} B$ if there is a computable f and a k so that $|D_{f(x)}| = k$ and

$$x \in A \iff D_{f(x)} \cap B \neq \varnothing.$$

(disjunctive): $A \leq_{\mathrm{d}} B$ if there is a computable f so that

$$x \in A \iff D_{f(x)} \cap B \neq \varnothing.$$

(conjunctive): $A \leq_{\mathrm{c}} B$ if there is a computable f so that

$$x \in A \iff D_{f(x)} \subseteq B.$$

(parity[2]): $A \leq_{\oplus} B$ if there is a computable f so that

$$x \in A \iff |D_{f(x)} \cap B| \equiv 1 \pmod 2$$

For later reference we call the expression $\chi_B(x_1) \oplus \ldots \oplus \chi_B(x_n)$ a parity condition (for B and $\{x_1, \ldots, x_n\}$). So we have equivalently,

$$x \in A \iff \text{the parity condition for } B \text{ and } D_{f(x)} \text{ evaluates to 1.}$$

(positive): $A \leq_{\mathrm{p}} B$ if there is a computable f and a computable $\alpha : \omega \times \{0,1\}^* \to \{0,1\}$ so that $\alpha_x(\cdot)$ is a *positive* function, i.e. $\alpha_x(\chi_S(D)) \leq \alpha_x(\chi_{S'}(D))$ for any finite set D and all $S \subseteq S'$, and

$$x \in A \iff \alpha_x(\chi_B(D_{f(x)})) = 1,$$

writing $\alpha_x(v)$ for $\alpha(x, v)$.

(truth-table): $A \leq_{\mathrm{tt}} B$ if there is a computable f and a computable $\alpha : \omega \times \{0,1\}^* \to \{0,1\}$ so that

$$x \in A \iff \alpha_x(\chi_B(D_{f(x)})) = 1,$$

writing $\alpha_x(v)$ for $\alpha(x, v)$.

(Q): $A \leq_{\mathrm{Q}} B$ if there is a computable f so that

$$x \in A \iff W_{f(x)} \subseteq B.$$

(weak truth-table): $A \leq_{\mathrm{wtt}} B$ if there is a Turing-reduction Φ^X so that $A = \Phi^B$ and the use of Φ^B on input x is computably bounded in x.

For more background on computability terminology and standard notation please check [21; 22; 27; 28].

3 Conjunctive, Disjunctive and Bounded Truth-Table Reductions

The following old result of Lachlan's (see [21, III.9.3] or [27, II.4.16]) is at the root of many of the truth-table results.

Proposition 3.1 (Lachlan [14]) *If $K \leq_{\mathrm{m}} A \times B$ and at least one of A and B is c.e., then either $K \leq_{\mathrm{m}} A$ or $K \leq_{\mathrm{m}} B$.*

Since simple sets cannot be m-complete and $A \oplus B \leq_{\mathrm{m}} A \times B$, this immediately implies that simple sets are not m-cuppable, and it is not much harder to obtain the same result for conjunctive reductions, as Degtev did in 1979 (see [22, page 603]).

Lemma 3.2 (Degtev [3]) *If $K \leq_{\mathrm{c}} A \oplus B$, and at least one of A and B is c.e. then either $K \leq_{\mathrm{c}} A$, or $K \leq_{\mathrm{c}} B$.*

The lemma immediately implies the following theorem, since hypersimple sets cannot be tt-complete let alone c-complete.

Theorem 3.3 (Degtev) *A hypersimple set is not c-cuppable.*

The following proof is taken from Odifreddi [22, Exercise X.7.23c].

Proof of Lemma 3.2 By assumption, there are computable functions f and g such that $x \in K$ if and only if $D_{f(x)} \subseteq A$ and $D_{g(x)} \subseteq B$. Let $E = \{x : D_{f(x)} \subseteq A\}$, and $F = \{x : D_{g(x)} \subseteq B\}$. Then at least one of E and F is c.e., and $K \leq_{\mathrm{m}} E \times F$, whence $K \leq_{\mathrm{m}} E$, or $K \leq_{\mathrm{m}} F$ by Lachlan's result. This implies the conclusion of the lemma. $\square$

Lachlan's result was elegantly restated by Kobzev.

Proposition 3.4 (Kobzev[12])

(*i*) *If A is productive and B is c.e., then either $A \cap B$ or $A \cap \overline{B}$ is productive.*
(*ii*) *If A is creative and B is computable, then either $A \cap B$ or $A \cap \overline{B}$ is creative.*

To deal with disjunctive reductions we need a strengthening of Kobzev's result: a uniform version of Proposition 3.4 (ii). Lachlan's proof, however, on which Kobzev's is based, uses a nonuniform proof: two strategies are pursued, one trying to build a reduction to A and the other a reduction to B. Furthermore the second strategy will only yield a reduction which is correct up to finitely many errors. Fortunately the first strategy yields a reduction uniformly, hence if we know that the first strategy succeeds we get the reduction uniformly.

Lemma 3.5 (Uniform Kobzev (creative sets)) *If A is creative and B is computable and we know that exactly one of $A \cap B$ and $A \cap \overline{B}$ is creative and which one it is, then we can find an m-reduction from K to that set uniformly in an m-reduction from K to A and a computable index of B.*

Proof Assume that exactly one of $A \cap B$ and $A \cap \overline{B}$ is creative and we know which one. Start the Lachlan proof with the uniform strategy working on the set we know to be creative. Since the other strategy has to fail (the other set not being creative) this will (uniformly) yield a reduction to the creative set. □

With this tool we can settle the disjunctive case, which is basically taken from [10; 26].

Lemma 3.6 *If A is a simple set, and $K \leq_{\mathrm{d}} A \oplus B$, then $K \leq_{\mathrm{d}} B$.*

Note that we cannot expect a result analogous to Lemma 3.2, since there are c.e. sets A and B so that $K \leq_{\mathrm{d}} A \oplus B$ without either $K \leq_{\mathrm{d}} A$, or $K \leq_{\mathrm{d}} B$ being true (just let A and B be a Friedberg splitting of K).

The lemma immediately implies the following theorem.

Theorem 3.7 *Simple sets are not d-cuppable.*

Proof of Lemma 3.6 Assume $K \leq_{\mathrm{d}} A \oplus B$. Then there are two computable functions f and g such that $x \in K$ iff $D_{f(x)} \cap A \neq \varnothing$ or $D_{g(x)} \cap B \neq \varnothing$.

For a set D define $E_D = \{x : D_{f(x)} \subseteq D\}$. We claim that there is a finite set $D \subseteq \overline{A}$ such that $K \cap E_D$ is creative. Since $x \in K \cap E_D$ iff $x \in E_D$ and $D_{g(x)} \cap B \neq \varnothing$ the claim implies that $K \leq_{\mathrm{m}} K \cap E_D \leq_{\mathrm{d}} B$ and we are done.

We are left with the verification of the claim. Assume for a contradiction that $K \cap E_D$ is not creative for any finite set $D \subseteq \overline{A}$. Because of Proposition 3.4, (ii) this means that $K \cap \overline{E_D}$ is creative for all finite subsets D of $\overline{A}$. By Lemma 3.5 we can even find a productive function for $\overline{K \cap \overline{E_D}}$ uniformly in the finite set D. Start with $F_0 = \varnothing$ and $C_0 = E_{F_0}$. Then C_0 is a c.e. set in the complement of $K \cap \overline{E_{F_0}}$, hence we can effectively find an element $y_0 \in \overline{K \cap \overline{E_{F_0}}} - C_0 = \overline{K} \cap \overline{E_{F_0}}$ using the productive function. For this element we have $D_{f(y_0)} \cap A = \varnothing$ and $D_{f(y_0)} \not\subseteq F_0 = \varnothing$. We repeat this procedure with $F_1 = D_{f(y_0)} \cup F_0$, $C_1 = E_{F_1}$, and so on. Because of the uniformity we get a c.e. set $\bigcup_{i=0}^{\infty} F_i$ which is a subset of $\overline{A}$, since all the F_i are, and infinite, since $F_i \subsetneq F_{i+1}$ for all i. This contradicts the simplicity of A. □

The result also implies that simple sets are not *btt*-cuppable in the c.e. degrees using a result of Lachlan [16] and Kobzev [12] that *btt*-complete sets are *bd*-complete and observing that in the proof of Lemma 3.6 the disjunctive reduction to B is bounded if the original reduction was. This result was first shown by the second author [26], and, independently, by Downey [7]. Downey's paper contains the stronger result, using a different proof, that simple sets are not *btt*-cuppable. We show how to extend the proof in [26] to obtain the same result.

Theorem 3.8 (Downey[8]) *Simple sets are not btt-cuppable.*

The theorem is an immediate consequence of the following lemma.

Lemma 3.9 *If A is simple, and $K \leq_{\mathrm{btt}} A \oplus B$, then $K \leq_{\mathrm{btt}} B$.*

Lachlan and Kobzev independently showed that *btt*-complete sets are *bd*-complete, the former using the technique of Proposition 3.1, the latter giving a proof using his result on productive sets. We will need a slight generalization of this result which we will prove following Kobzev's ideas.

Lemma 3.10 *If $K \leq_{\mathrm{btt}} A \oplus B$, and A is c.e., then there are computable functions f and g and a truth-table α such that*

$$x \in K \quad \leftrightarrow \quad [D_{f(x)} \cap A \neq \varnothing] \vee [\alpha(\chi_B(D_{g(x)})) = 1]$$

where $|D_{f(x)}| \leq k$ for all x, and some $k \in \omega$.

Taking B to be the empty set implies the original result by Lachlan and Kobzev.

Proof Let $K \leq_{\mathrm{btt}} A \oplus B$. We can assume that the reduction uses a fixed truth-table (Fischer, see [21, III.8.6]): $x \in K$ if and only if

$$\alpha(\chi_A(D_{f(x)}), \chi_B(D_{g(x)}))) = 1$$

where f and g are computable functions generating the queries to A and B respectively, and $|D_{f(x)}| = k_1$, $|D_{g(x)}| = k_2$ for some fixed $k_1, k_2 \in \omega$ and all x. We will show by induction on $k = k_1 + k_2$ that there are functions f' and g' such that

$$x \in K \quad \leftrightarrow \quad [D_{f'(x)} \cap A \neq \varnothing] \vee [\alpha(\chi_B(D_{g'(x)})) = 1] \tag{1}$$

If $k = 1$, then either $K \leq_{\mathrm{m}} A \oplus B \leq_{\mathrm{m}} A \times B$, or $\overline{K} \leq_{\mathrm{m}} A \oplus B$. In the first case Lachlan's result implies that either $K \leq_{\mathrm{m}} A$ or $K \leq_{\mathrm{m}} B$ both of which are subsumed by Schema (1). The second case implies $K \leq_{\mathrm{m}} \overline{B}$ since A is c.e.

Assume $k > 1$. Let $C = \{x : D_{f(x)} \cap A \neq \varnothing\}$ (which is a c.e. set). By Proposition 3.4 there are two cases.

(*i*) $\overline{K} \cap \overline{C}$ is productive. Then $K \cup C$ is creative, and $x \in K \cup C$ if and only if

$$[\alpha(\chi_A(D_{f(x)}), \chi_B(D_{g(x)})) = 1] \vee [D_{f(x)} \cap A \neq \varnothing].$$

This is equivalent to

$$[\alpha(\chi_A(D_{f(x)}), \chi_B(D_{g(x)})) = 1 \wedge D_{f(x)} \cap A = \varnothing] \vee [D_{f(x)} \cap A \neq \varnothing],$$

or, in other words

$$[\alpha(0^{k_1}, \chi_B(D_{g(x)})) = 1] \vee [D_{f(x)} \cap A \neq \varnothing].$$

Since $K \cup C$ is creative we can define f' and g' as required in Schema (1).

(*ii*) $\overline{K} \cap C$ is productive. Since C is c.e. there is a computable function h such that $C = h(\omega)$. Let $K_1 = h^{-1}(K \cap C)$. Then K_1 is creative (using that $\overline{K} \cap C$ is productive). We have $x \in K_1$ if and only if $h(x) \in K$ (if and only if $h(x) \in K \cap C$). By definition, $h(x) \in C$; that is, $D_{f(h(x))} \cap A \neq \varnothing$ for all x.

Define

$$\begin{aligned} R_i \quad = \quad & \{x : D_{f(h(x))} = \{y_1 < \cdots < y_k\}, \text{ and } y_i \text{ is the first} \\ & \quad \text{element of } D_{f(h(x))} \text{ enumerated in } A\}, \end{aligned}$$

for $(1 \leq i \leq k)$. The R_i are a computable partition of ω, hence by Proposition 3.4, (ii) there is an i such that $K_1 \cap R_i$ is creative. Moreover, $x \in K_1 \cap R_i$ if and only if

$$\alpha'(\chi_A(D_{f'(x)}), \chi_B(D_{g'(x)})) = 1,$$

where we obtain α' from α by setting the i-th input to one, and $D_{f'(x)} = D_{f(h(x))} - \{y_i\}$ for $D_{f(h(x))} = \{y_1 < \cdots < y_k\}$, and $g'(x) = g(h(x))$. Since $|D_{f'(x)}| + |D_{g'(x)}| < k$, we can now apply the induction hypothesis to obtain the result.

□

With this we are now ready to prove Lemma 3.9. The proof is virtually identical to the proof of Lemma 3.6.

Proof of Lemma 3.9 Suppose $K \leq_{\text{btt}} A \oplus B$ where A is simple. By Lemma 3.10 we know that there are computable functions f and g such that $x \in K$ if and only if $D_{f(x)} \cap A \neq \varnothing$ or $\alpha(\chi_B(D_{g(x)})) = 1$.

For a set D define $E_D = \{x : D_{f(x)} \subseteq D\}$. We claim that there is a finite set $D \subseteq \overline{A}$ such that $K \cap E_D$ is creative. Since $x \in K \cap E_D$ if and only if $x \in E_D$ and $\alpha(\chi_B(D_{g(x)})) = 1$ the claim implies that $K \leq_{\text{m}} K \cap E_D \leq_{\text{btt}} B$ and we are done.

Assume for a contradiction that the claim is wrong, and $K \cap E_D$ is not creative for any finite set $D \subseteq \overline{A}$. Because of Proposition 3.4 this means that $K \cap \overline{E_D}$ is creative for all finite subsets D of $\overline{A}$. By Lemma 3.5 we can even find a productive function for $\overline{K \cap \overline{E_D}}$ uniformly in the finite set D. Start with $F_0 = \varnothing$ and $C_0 = E_{F_0}$. Then C_0 is a c.e. set in the complement of $K \cap \overline{E_{F_0}}$, hence we can effectively find an element $y_0 \in \overline{K \cap \overline{E_{F_0}}} - C_0 = \overline{K} \cap \overline{E_{F_0}}$ using the productive function. For this element we have $D_{f(y_0)} \cap A = \varnothing$ and $D_{f(y_0)} \not\subseteq F_0 = \varnothing$. We repeat this procedure with $F_1 = D_{f(y_0)} \cup F_0$, $C_1 = E_{F_1}$, and so on. Because of the uniformity we get a c.e. set $\bigcup_{i=0}^{\infty} F_i$ which is a subset of $\overline{A}$, since all the F_i are, and infinite, since $F_i \subsetneq F_{i+1}$ for all i. This contradicts the simplicity of A. □

4 Positive Reductions

Theorem 4.1 *Hypersimple sets are not p-cuppable in the c.e. degrees.*

We will prove the theorem in the following form.

Lemma 4.2 *If A is hypersimple, B is c.e., and $K \leq_{\text{p}} A \oplus B$, then $K \leq_{\text{p}} B$.*

For the proof we will use a uniform version of Kobzev's result for productive sets. Note that this result is not symmetric: the proof would not yield a reduction to $A \cap \overline{B}$ uniformly.

Lemma 4.3 (Uniform Kobzev (productive sets)) *If A is productive, B is c.e. and we know that $A \cap \overline{B}$ is not productive, then $A \cap B$ is productive, and an m-reduction from $\overline{K}$ to $A \cap B$ can be found uniformly in an m-reduction from $\overline{K}$ to A and an index of B.*

Proof As above we use Lachlan's proof, and start it with the two sets $\overline{A} \cup \overline{B}$ and $\overline{A} \cup B$. The strategy for reducing K to $\overline{A} \cup B$ must fail (since $A \cap \overline{B}$ is not productive), and hence the first strategy, namely the one reducing K to $\overline{A} \cup \overline{B}$ must succeed. We only need to observe that the resulting reduction can be determined uniformly in the index of the reduction from K to $\overline{A}$ and an index of B. □

With this we can complete the proof.

Proof of Lemma 4.2 Assume $K \leq_{\mathrm{p}} A \oplus B$. Then there are computable functions f, g, and α_x such that

$$x \in K \quad \Leftrightarrow \quad \alpha_x(\chi_A(D_{f(x)}), \chi_B(D_{g(x)})) = 1,$$

and α_x is positive. Let $S_n = A \cup \{x : x \geq n\}$, and define

$$E_n = \{x : \alpha_x(\chi_{S_n}(D_{f(x)}), \chi_B(D_{g(x)})) = 1\}.$$

The set E_n is c.e. (uniformly in n) since B is. There are two cases: (i) there is an n such that $\overline{K} \cap \overline{E_n}$ is productive, or (ii) $\overline{K} \cap E_n$ is productive for all n (uniformly in n by Lemma 4.3).

In case (i) we note that

$$\begin{aligned} x \in \overline{K} \cap \overline{E_n} \quad &\Leftrightarrow \quad \alpha_x(\chi_A(D_{f(x)}), \chi_B(D_{g(x)})) = 0 \\ &\qquad \wedge\, \alpha_x(\chi_{S_n}(D_{f(x)}), \chi_B(D_{g(x)})) = 0 \\ &\Leftrightarrow \quad \alpha_x(\chi_{S_n}(D_{f(x)}), \chi_B(D_{g(x)})) = 0. \end{aligned}$$

The last equivalence is true because $\alpha_x(\chi_{S_n}(D_{f(x)}), \chi_B(D_{g(x)})) = 0$ implies $\alpha_x(\chi_A(D_{f(x)}), \chi_B(D_{g(x)})) = 0$ (since $A \subseteq S_n$, and α_x is positive). Since $\overline{K} \cap \overline{E_n}$ is productive this shows that $K \leq_{\mathrm{p}} B$ (S_n is computable).

In case (ii) we can effectively in n find an element $h(n)$ of $\overline{K} \cap E_n$. Note that

$$\begin{aligned} x \in \overline{K} \cap E_n \quad &\Leftrightarrow \quad \alpha_x(\chi_A(D_{f(x)}), \chi_B(D_{g(x)})) = 0 \\ &\qquad \wedge\, \alpha_x(\chi_{S_n}(D_{f(x)}), \chi_B(D_{g(x)})) = 1. \end{aligned}$$

This means that A and S_n must differ on $D_{f(x)}$, or in other words $D_{f(h(n))} \cap \{m : m \geq n\} \cap \overline{A} \neq \varnothing$. Hence A is not hypersimple. $\square$

Remark 4.4 In Theorem 6.4 we will show how to construct hypersimple set H and a 2-c.e. set A such that $K \leq_{\mathrm{d}} A \times H$ but not $K \leq_{\mathrm{p}} A$. That is, positive hypersimple degrees can be p-cuppable. However, as we will see in Corollary 6.8, dense simple sets are not p-cuppable.

5 Parity Reductions

We know that hypersimple sets cannot be tt-complete. However, as we will see in Section 6 they may be tt-cuppable, even in the c.e. degrees. The next result shows that this does not happen if we sharpen tt-reducibility to parity-reducibility.

Theorem 5.1 *A hypersimple set is not $\oplus$-cuppable.*

Proof

Let A be hypersimple and assume that $K \leq_{\oplus} A \oplus B$ where B is any, not necessarily c.e., set. We show that $K \leq_{\oplus} B$.

Uniformly, for any given x, we construct a c.e. set B_x and a sequence of disjoint finite intervals I_n^x, $n \in \omega$, inductively as follows. By the recursion theorem, we may assume that we have a parity reduction Φ_x from B_x to $A \oplus B$.

Suppose we have constructed intervals $I^x_m, m < n$. Let k be their maximum element. Choose $2^{k+1} + 1$ new numbers and compute the maximum element k' used by Φ_x when applied to any of the new numbers. Let $I^x_m = [k + 1, k']$.

Simultaneously, we enumerate A until we find an interval I' that is contained in A. Such an interval has to exist since otherwise each interval intersects $\overline{A}$ contradicting the hypersimplicity of A.

Suppose we find $I' = [k+1, k'] \subseteq A$. Consider the $2^{k+1}+1$ numbers chosen in the construction of this interval. Each such number i gives rise to a parity condition $p_i = \Phi_x(i)$. Then there must be two distinct numbers $a < b \leq k$ whose parity conditions agree on queries to A which are at most k. Moreover, the queries to the A-part which are larger than k can be evaluated since each such query is answered in the positive.

Now do the following: If $x \in K$ we enumerate a into B_x, otherwise $B_x = \varnothing$.

Then $x \in K$ iff the joined parity condition $p_a \oplus p_b$ evaluates to 1. But in this condition all queries to A cancel out or can be evaluated. Therefore, we have in fact a parity reduction to B. □

6 Truth-table Reductions

6.1 The Case of Hypersimple Sets The main open cuppability problem is that for truth-table reductions. Unexpectedly, there are hypersimple sets that *are* tt-cuppable, even in the c.e. degrees. Such sets have to be wtt-complete, by the result of Downey and Jockusch [9]. This explains, why the proof is rather involved: Lachlan's separation of wtt-completeness from tt-completeness was the most intricate proof for separating completeness notions [16] (or [21, Theorem III.9.1]).

Theorem 6.1 *There is a hypersimple set that is tt-cuppable in the c.e. degrees.*

Proof By a finite injury argument we construct c.e. sets A, B and a c.e. co-retraceable set H such that $K \leq_{\mathrm{tt}} A \oplus H$ and $B \not\leq_{\mathrm{tt}} A$.
Partition ω into an ascending sequence of intervals $I_x, x \geq 0$, such that

$$|I_x| = x + 1.$$

We will have $0 \notin H$. The tt-reduction Φ of K to $A \oplus H$ is defined as follows:

$$x \in K \iff \text{the } n\text{-th element of } I_x \text{ is in } A \text{ where } n = \max\{y \leq x : y \notin H\}.$$

Define a function $f : \omega \to \omega$ as follows: $f(0) = 0, f(\ell+1) = f(\ell)+2(\ell+2)+1$. W.l.o.g. we may assume that $K \subseteq \mathrm{range}(f)$.

Construction of A, B, H:

The construction proceeds in stages. At the beginning of stage $s + 1$ we enumerate for each $x \in K_s$ with $x < s$ the n-th number of I_x into A, where $n = \max\{y \leq x : y \notin H_s\}$. In this way we will have satisfied in the end that $K \leq_{\mathrm{tt}} A \oplus H$ via Φ.

For each i we will satisfy the following requirement R_i:

(R_i) If φ_i is total then $B \not\leq_{\mathrm{tt}} A$ via the tt-reduction coded by φ_i.

Each i is in exactly one of the states SETUP, WAIT, or PLAY. At the beginning of the construction all i are in state SETUP.

We say that i *requires attention* at stage $s+1$ if one of the following conditions holds.

a.) i is in state SETUP.

b.) i is in state WAIT and $\varphi_{i,s}(x_i)$ is defined where x_i is the current diagonalization witness of i.

c.) i is in state PLAY and a number less than or equal to the use of the tt-operator coded by $\varphi_i(x_i)$ was enumerated into A since the last time when i received attention.

At stage $s+1$ the least i that requires attention *receives attention.* Then the following is done according to in which state i is.

a.) If i is in state SETUP let ℓ be minimal such that $\{x : x > f(\ell)\} \cap H_s = \varnothing$ and $\ell > \ell_j$ for all $j < i$. We let $\ell_i = \ell$. Allocate the numbers in the interval $T_i = \{x : f(\ell_i) < x < f(\ell_i + 1)\}$ to i and choose a new unused diagonalisation witness x_i. Put all $j > i$ into state SETUP and put i into state WAIT.

b.) If i is in state WAIT and $\varphi_{i,s}(x_i)$ is defined let u_i be the maximal use of the associated tt-operator $\sigma_i^X(x_i)$. Find u' such that $u_i \in I_{u'}$. Dump all z with $f(\ell_i + 1) \leq z \leq \max(H_s \cup \{u'\})$ into H.

Let b_i be the greatest element of T_i that does not belong to H_s, and let $a_i = b_i - 1$. Consider the game on $\sigma_i^X(x_i)$ where initially $X = A_s$ and where player 1 may enumerate the a_i-th element of I_x into X and player 2 may enumerate the b_i-th element of I_x into X for any x with $f(\ell_i + 1) \leq x \leq u'$. After ω rounds player 1 wins iff $\sigma_i^X(x_i) = 0$.

Since this is essentially a finite game, it is determined and we can determine effectively which player has a winning strategy. Also this strategy is effective.

If player 1 has a winning strategy, we choose it, let $v_i = 0$, and enumerate x_i into B.

If player 2 has a winning strategy (i.e., he has a strategy that achieves $\sigma_i^X(x_i) = 1$ in the end) choose it and let $v_i = 1$. Then dump all $x \geq b_i, x \leq \max(H_s)$ into H and put all $j > i$ into state SETUP.

Enumerate elements into A according to the chosen strategy such that $\sigma_i^{A_{s+1}}(x_i) = v_i$.

Put i into state PLAY.

c.) If i is in state PLAY and elements less than or equal to u_i have been enumerated into A since the last time it received attention, do the following: Let x be the minimum of the enumerated elements. We distinguish two cases:

(1) If $x \leq \max(I_{f(\ell_i)})$, then dump all $z \geq a_i, z \leq \max(H_s)$ into H, cancel the witness of i and choose an unused new one, and set the state of i equal to WAIT and the state of all $j > i$ equal to SETUP.

(2) Otherwise, by the assumption on K, $x \geq \min(I_{f(\ell_i+1)})$. Then each of the enumerated elements x' must be the b_i-th element of $I_{x'}$, if $v_i = 0$, and the a_i-th element of $I_{x'}$, if $v_i = 1$. Now

enumerate elements into A according to the chosen strategy such that $\sigma_i^{A_{s+1}}(x_i) = v_i$. Keep the state of i equal to PLAY.

End of construction.

Verification:

Lemma 1: H is co-retraceable.
Proof: This follows from dumping.

Lemma 2: $K \leq_{\mathrm{tt}} A \oplus H$.
Proof: By the primary action taken in each stage $s+1$ we get:

$$x \in K \Rightarrow \Phi^{A \oplus H}(x) = 1.$$

For the other direction note that when we enumerate for some i in state PLAY at stage s' a number y, which is the n-th element of I_x, into A according to the chosen strategy, then n is not the greatest element of $\overline{H}_s \cap [0, \ldots, x]$, for all $s \geq s'$:

If $v_i = 0$, then y is the a_i-th element of I_x and the $b_i = a_i + 1$-th element is greater and does also not belong to $H_{s'}$. If later this element is enumerated into H, then the a_i-th is also enumerated into H (this can only happen by a dumping action of some $j \leq i$).

If $v_i = 1$, then y is the b_i-th element of I_x. But in that case we already enumerated b_i into H when we determined the strategy. Thus the assertion holds trivially.

Lemma 3: Each R_i is satisfied. Therefore $K \not\leq_{\mathrm{tt}} A$.
Proof: This follows by induction on i.

First, it is easy to see that in each stage s the intervals T_i, if defined, form an ascending sequence in i.

Second, we show by simultaneous induction on i the following statements:

(1) i is only finitely often in state SETUP.

(2) i is either almost always in state WAIT (then φ_i is not total), or almost always in state PLAY.

(3) i receives attention only finitely often.

(4) R_i is satisfied.

Suppose this holds for all $j < i$ and consider a stage s' such that no $j < i$ requires attention for all $s \geq s'$. Then at the latest at stage s', i is put into state WAIT with some witness x_i and an interval T_i. Let $\ell = \ell_i$, which does not change from this stage on.

If $\varphi_i(x_i)$ is not defined then i will stay in state WAIT in all of the following stages and will never require attention any more.

If $\varphi_i(x_i)$ is defined, then i will reach the state PLAY. Since there may be at most $\ell + 1$ numbers of K that may cause that a new witness is chosen, i switches back to WAIT at most $\ell + 1$ times. If φ_i is defined for all of these new witnesses, then i will end up in state PLAY and never leave it. Otherwise it will end up in state WAIT, but this means that φ_i is not total.

Therefore we have shown (1) and (2). Also clearly (3) holds if i ends up in state WAIT.

Now suppose i ends up in state PLAY and consider s'' such that, at stage $s''+1$, i enters state WAIT from state SETUP for the last time. Then T_i consists of $2(\ell+2)$ numbers all in $\overline{H}_{s''+1}$. Each time when at some later stage i leaves PLAY and enters WAIT, this happened since a number $f(k), k \leq \ell$ was enumerated into K (which caused enumerations into A by the global *tt*-reduction Φ and maybe additionally by the strategy of some R_j with $j < i$).

Clearly this happens at most $\ell + 1$ times. Each time the two greatest elements of T_i are enumerated into H. Since $|T_i| = 2(\ell+2)$, there will be at least 2 elements from $\overline{H}$ in T_i left. Thus the numbers $a_i, b_i \in T_i$ will always be defined when the state changes from WAIT to PLAY.

Now consider the final change from WAIT to PLAY. From then on we will always work on a fixed *tt*-operator $\sigma_i^X(x_i)$ for the final witness x_i. Clearly A changes below the use only finitely often. Therefore, (3) holds. Since A will never change below $I_{f(\ell+1)}$ our strategy ensures that $\sigma_i^A(x_i) = v_i \neq B(x_i)$:

Note that, by the dumping when we were in WAIT for the last time, for any $x \geq f(\ell+1)$ with $\max(I_x)$ less or equal than the use of $\sigma_i^X(x_i)$ we have that $\max(\overline{H} \cap [0, x])$ equals b_i, if $v_i = 0$ and equals a_i, if $v_i = 1$. Therefore the numbers that are forced into such an interval by the *tt*-reduction Φ are always equal to a legal move of the opponent in the corresponding game. The strategy is chosen in such a way that these moves are countered by additional numbers we enumerate into A such that $\sigma_i^{A_s}(x_i) = v_i$ after each such stage s and in the end.

Thus, B is not *tt*-reducible by the *tt*-reduction coded by φ_i and R_i is satisfied, i.e., (4) holds.

Lemma 4: H is hypersimple.
Proof: From Lemmas 2, 3 it follows that H is noncomputable. Since every c.e., noncomputable co-regressive set is hypersimple ([4; 5] or [21, Ex. III.3.14a]), Lemma 1 implies that H is hypersimple. □

Remark 6.2 (1) The idea of using games and strategy-stealing to diagonalize a *tt*-operator appears first in [13] and later in [19]. There it was used in the realm of Kolmogorov complexity to construct certain *wtt*-complete but not *tt*-complete sets. Such sets were first constructed by Lachlan [16] using a different method that is less flexible.

(2) Since there are c.e., semirecursive sets S for which $K \leq_{\mathrm{tt}} S$, the requirement $K \leq_{\mathrm{tt}} \Phi^{A \oplus H}$ in the proof of Theorem 6.1 can alternatively also be written as $S \leq_{\mathrm{tt}} \Phi^{A \oplus H}$.

This allows us to assume that elements get enumerated into the I_x as initial segments of a computable linear ordering. This simplifies the presentation of the enumeration game played by the two players to the following game: given a number n, the two players alternately play a number $x < n$, where each player has to play numbers in nondecreasing order. The positions of the game are pairs (x, y) with x being the current value of player 1 and y being the current value of player 2. A particular pair (x, y) is either a 1-*position* or a 2-*position.* The initial position is $(0, 0)$. The legal moves of player i are moves to an i-position

($i = 1, 2$). A player loses if he does not have a legal move when it is his turn to move.

The game corresponds to enumerating increasing initial segments of an n-element linear ordering. The tt-operator determines which positions are 1-positions.

For this game it is easy to decide which player has a winning strategy by having both players follow this strategy: in position (a, b) if the position is a 1-position and player 1 moves or a 2-position and player 2 moves, do not do anything; otherwise pick the smallest number $x > a$ (player 1) or $x > b$ (player 2) such that (x, b) is a 1-position (or (a, x) is a 2-position if player 2 moves). If there is no such move, the active player has lost.

The player who wins this simulation also has a winning strategy for the original game, i.e. the winning strategy is to extend the initial segments as conservatively as possible (and necessary).

This simplifies the finite game that has to be considered in the proof of Theorem 6.1, but it does not simplify the proof.

(3) Downey [6] gave a general construction of c.e., noncomputable non-tt-cuppable sets. His proof can be adapted to show that such sets can be co-retraceable. Thus our result cannot be generalized to arbitrary c.e., noncomputable, co-retraceable sets.

(4) It is easy to make H low by adding standard lowness requirements or Turing-complete, by encoding the Halting Problem. Are there c.e., noncomputable Turing degrees that do not contain tt-cuppable hypersimple sets? This question seems to be hard and appears to require new techniques.

The proof of Theorem 6.1 can be adapted to yield an r-maximal instead of a co-retraceable set.

Theorem 6.3 *There is an r-maximal set H that is tt-cuppable in the c.e. degrees.*

Sketch of Proof For this proof we combine (a) Lachlan's construction of an r-maximal set which is not dense simple (see [15], [22, pages 393 f.] or [27, X.5.10]) with (b) our construction of a hypersimple tt-cuppable set.

The construction in (a) uses a sequence of intervals $(T_n)_{n\in\omega}$ which can be used for diagonalization as in (b). To this end we associate with each interval that has not been completely enumerated into H a tt-operator σ_i; we can arrange things so that σ_i is associated with the i-th interval which has not been completely enumerated into H; in other words, an interval T_n might be associated in turn with $\sigma_{i_1}, \sigma_{i_2}$, etc. where $n > i_1 > i_2 \ldots$; this works if the interval is chosen large enough. Once the interval does not change anymore, as part of (a), the diagonalization in part (b) will be successful. If the interval changes, we choose a new witness and enumerate the a_i-th and the b_i-th element of the interval into H as in the previous proof. $\square$

The following two results are rather conventional diagonalization arguments that are simplified versions of the construction of Theorem 6.1. We therefore provide only proof sketches.

We next make good on our promise from Section 4 by showing how to modify the proof of Theorem 6.1 to work for positive reductions.

Theorem 6.4 *There is a hypersimple set H and a 2-c.e. set A such that $K \leq_{\mathrm{d}} A \times H$ but $K \not\leq_{\mathrm{p}} A$.*

Sketch of Proof We are using the setup of the proof of Theorem 6.1 and define

$$x \in K \iff \text{there is an } n < |I_x| \text{ such that the } n\text{-th element of } I_x \text{ belongs to } A \text{ and } n \in H.$$

In this way we satisfy $K \leq_{\mathrm{d}} A \times H$.

To diagonalize the positive *tt*-operator σ_i^X we have numbers in the interval T_i to play with. Let $b_i = \min(T_i \cap H_s)$ and $a_i = b_i - 1$. Then we check whether we can enumerate numbers into X which are the a_i-th numbers of I_x for x with $f(\ell+1) \leq x \leq u'$ such that $\sigma_i^X(x_i) = 1$.

If yes, we enumerate these numbers into A and keep x_i out of B. This achieves $B(x_i) = 0 \neq 1 = \sigma_i^A(x_i)$. Note that since the *tt*-operator is positive it will stay at 1 even if additional elements are enumerated for the sake of the global requirement $K \leq_{\mathrm{d}} A \oplus H$.

If no, we enumerate x_i into B, a_i into H, and put the a_i-th element of I_x into H if x with $f(\ell+1) \leq x \leq u'$ is enumerated into K. Then we know that $\sigma_i^A(x_i) = 0 \neq 1 = B(x_i)$.

Note that in this case the game degenerates to a 1-mover. If the setup of the i-th diagonalization is cancelled by the action of some $j < i$ and the yes-case has happened, we take back the numbers we enumerated into A. This makes A 2-c.e. □

The next result shows that the non-cuppability result of Downey and Jockusch for *computably enumerable* degrees cannot be strengthened.

Theorem 6.5 *There is a hypersimple set H and a 2-c.e. set A such that $K \leq_{\mathrm{tt}} A \oplus H$ but $K \not\leq_{\mathrm{wtt}} A$.*

Sketch of Proof Again we use the setup of Theorem 6.1 . This time it is convenient to replace K by a c.e. semirecursive set $S \equiv_{\mathrm{tt}} K$. Let S be the lower cut of a computable linear ordering $\prec$. We will make sure that

$$x \in S \iff \begin{array}{l} \text{(a) the } n\text{-th element of } I_x \text{ is even and belongs to } A, \text{ or} \\ \text{(b) the } n\text{-th element of } I_x \text{ is odd and does not belong to } A, \\ \text{where } n = \max\{y \leq x : y \notin H\}. \end{array}$$

To diagonalize the *wtt*-operator σ_i^X we are playing with the numbers in T_i. Let $b_i = \max(T_i \cap \overline{H_s})$ and $a_i = b_i - 1$. Furthermore we may assume that b_i is even.

Let $z_1 \prec \ldots \prec z_k$ be the linear ordering of all x with $f(\ell+1) \leq x \leq u'$. Let $y_1, \ldots, y_k$ be the a_i-th elements of $I_{z_1}, \ldots, I_{z_k}$, respectively. We enumerate the y_i's one by one in the inverse ordering (starting with y_k) into A and wait until $\sigma_i^A(x_i)$ is defined and equal to 0, after each new element has been enumerated.

If at one point we wait forever, either $\sigma_i^A(x_i) = 1 \neq 0 = B(x_i)$ or $\sigma_i^A(x_i)$ is undefined. In either case σ_i^X is diagonalized.

So suppose the *wtt*-operator is always defined and equal to 0. Then we enumerate x_i into B, b_i into H and reset the enumeration of A to a previous stage as long as some elements z_j are enumerated into S. Since we can assume that the z_j's are enumerated according to the ordering $\prec$, the y_j are extracted in the reverse ordering as they have been enumerated (in addition we may have to extract some of the b_i-th elements of the I_x to restore a previous stage of A). This makes sure that we always have $\sigma_i^A(x_i) = 0 \neq 1 = B(x_i)$ and A is 2-c.e. □

6.2 The Case of Dense Simple Sets In the previous section we saw that hypersimple sets may be *tt*-cuppable. For sets with thinner complements this is no longer true as we will see presently.

A set A is *dense immune* if the function p_A enumerating it in order dominates every total computable function. A is *dense simple* if it is c.e. and its complement is dense immune. Any dense simple set is hypersimple. On the other hand, hyperhypersimple sets, and thus maximal sets, are dense simple.

We will use Robinson's result [25] that a set A is dense immune if and only if for every disjoint strong array $(V_i)_{i\in\omega}$ there is an m such that $|V_n \cap A| < n$ for all $n \geq m$ (see [27, XI.1.10]).

Theorem 6.6 *A dense simple set is not tt-cuppable.*

Proof Suppose M is dense simple and $K \leq_{\mathrm{tt}} A \oplus M$ via a *tt*-reduction Φ. Applying a clever idea of Downey [6, Theorem 2.1] we will show that $M \leq_{\mathrm{tt}} A$, and, thereby, $K \leq_{\mathrm{tt}} A$.

We construct a c.e. set B as follows. By the recursion theorem, we can assume that we have an m-reduction f from B to K. Partition ω into a sequence of intervals $I_0, I_1, I_2, \ldots$ with $|I_n| = n$. Let $m_0 = 0$ and let m_{n+1} be the maximum of $m_n + 1$ and the largest element used by Φ on any input from $f(I_n)$, and define $J_n = \{x : m_n < x \leq m_{n+1}\}$.

For every n, enumerate $M = \cup_{s=0}^{\infty} M_s$ until $J_{n+1} \cap \overline{M_s}$ contains less than $n+1$ elements $x_1, \ldots, x_k$ $(k \leq n)$. Pick k corresponding elements $y_1, \ldots, y_k$ in I_n and enumerate y_i in B if and only if x_i in M. This concludes the construction of B.

We claim that $M \leq_{\mathrm{tt}} A$. Since M is dense simple we know that

$$|J_n \cap \overline{M}| < n$$

for all $n \geq a$ (for some a). We define a *tt*-reduction from M to A inductively (following the idea of Downey). For every element $x \in \cup_{i\leq a} J_i$ pick a *tt*-reduction from M to A. For $x \in J_{n+1}$ $(n+1 > a)$ proceed as follows: Simulate the construction of B until J_{n+1} contains less than $n+1$ elements of $\overline{M_s}$. By the choice of a, the simulation terminates. If $x \in M_s$ let the reduction map to true. Otherwise, x equals one of the x_i and is associated with some $y_i \in I_n$. By construction $x \in M$ iff $y_i \in B$ iff $f(y_i) \in K$ iff $\Phi^{A\oplus M}(f(y_i)) = 1$. The use of Φ on input $f(y_i)$ is bounded by m_{n+1}. By the inductive assumption we have *tt*-reductions for any $x' \leq m_{n+1}$ from M to A. Hence we can we can rebuild the *tt*-reduction for $f(y_i)$ replacing every query to M by queries to A. In this way we obtain a *tt*-reduction for x from M to A. □

Note that virtually the same proof holds for *wtt*-reductions.

Corollary 6.7 *A dense simple set is not wtt-cuppable.*

The proof also works for positive reductions, since the rebuilt tt-reduction is positive if Φ and the reductions from the inductive assumption are positive.

Corollary 6.8 *A dense simple set is not p-cuppable.*

6.3 The Case of Semirecursive, η-Maximal Sets In Theorem 6.6 we saw that dense simple sets are not tt-cuppable, and therefore maximal sets are not tt-cuppable. This remains true if we relax maximality to η-maximality assuming the set in question is c.e. and semirecursive, a natural assumption in light of Degtev's and Marchenkov's solution to Post's problem.

Theorem 6.9 *A c.e., semirecursive and η-maximal set is not tt-cuppable.*

Lemma 6.10 *Assume A is c.e., noncomputable, semirecursive and η'-maximal. Then there is a (coarser) positive equivalence relation η such that A is η-maximal and for any strong array $(J_n)_{n\in\omega}$ that partitions ω, and $P_n := \{[x]_\eta : x \in J_m, m < n\}, n \in \omega$, there is a finite set Q of η-equivalence classes from $\overline{A}$ such that*

$$|\{\ [x]_\eta : x \in J_n \cap \overline{A} \text{ and } [x]_\eta \notin P_n \cup Q\ \}| \leq 1$$

for all n.

We can modify the proof of Theorem 6.6 to apply to semirecursive η-maximal sets by replacing Robinson's result with our Lemma 6.10. As in Theorem 6.6, the proof can also be made to work for p- or wtt-reductions.

Proof of Lemma 6.10 Since A is semirecursive, it is the lower cut of a computable linear ordering [21, Proposition III.5.4] (attributed to McLaughlin and Appel by [11]). Assume A is η'-maximal. We replace η' by the transitive and symmetric closure η of the relation $x\eta''y$ defined as

$$x\eta'y \vee (\exists z)[x \prec y \prec z \wedge x\eta'z].$$

(This idea is due to Kobzev as described in [22, p. 595].) It follows that η is a positive equivalence relation (coarser than η') and A is η-closed. Furthermore A is η-maximal and the linear ordering is compatible with η, i.e. for all x, y, z

$$x\eta z \wedge x \prec y \prec z \Rightarrow x\eta y.$$

Define $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ as follows:

$$\begin{aligned} b_0 &= \min\{n : J_n \cap \overline{A} \neq \varnothing\} \\ a_i &= \min_{\prec}\{J_{b_i} \cap \overline{A}\} \qquad \text{(the minimum with respect to } \prec) \\ b_{i+1} &= \min\{n > b_i : \text{there is an } a \in J_n \cap \overline{A} \text{ with } a \prec a_i\}. \end{aligned}$$

Both sequences are well-defined, since A is not computable. Furthermore, $(a_i)_{i\in\omega}$ is strictly decreasing w.r.t. $\prec$. There is a c.e. set $B \supseteq A$ such that

$$B \cap \overline{A} = \cup_{i\in\omega}[a_i]_\eta.$$

(This is easily done by approximating a_i using approximations $\overline{A_s}$ instead of $\overline{A}$.) Since A is η-maximal, we can conclude that either $B \cap \overline{A}$ or $\overline{B} \cap \overline{A}$ is η-finite. If $B \cap \overline{A}$ were η-finite, then $\overline{A} = \{[b]_\eta : [a]_\eta \prec [b]_\eta \text{ and } [a]_\eta \subseteq B \cap \overline{A}\}$

would be c.e., contradicting the assumption. We conclude that $\overline{B} \cap \overline{A}$ is η-finite, i.e., $Q := \{[x]_\eta : x \in \overline{B} \cap \overline{A}\}$ is finite. Assume $[J_n \cap \overline{A}]_\eta$ contains exactly k equivalence classes $[x_1]_\eta \prec \ldots \prec [x_k]_\eta$ with representatives from J_n and choose i minimal such that $b_i \geq n$. Then $a_i \preceq x_1$ by definition of a_i. Any $[x_j]_\eta$ $(j \geq 2)$ that does not occur among $[a_1]_\eta, \ldots, [a_i]_\eta$ has empty intersection with B and therefore occurs in Q. Hence $[x_1]_\eta$ is the only (potentially) new η-equivalence class in $J_n \cap \overline{A}$. This proves the lemma. □

7 Q- and T-cuppability

Post's original program of finding structural properties that force Turing-incompleteness was completed twenty years after the priority constructions of Friedberg and Muchnik using η-maximality and the relationship between Tennenbaum's Q- and Turing-reductions for semirecursive sets. While there is a strong non-cuppability result for Q-reductions, η-maximality and semirecursiveness are not strong enough to force non-cuppability with respect to Turing reductions.

Theorem 7.1 *If $K \leq_{\mathrm{Q}} A \oplus B$ then $K \leq_{\mathrm{Q}} A$ or $K \leq_{\mathrm{Q}} B$.*

Proof Suppose $K \leq_{\mathrm{Q}} A \oplus B$. There are computable functions g, h such that for all x:

$$x \in K \;\Leftrightarrow\; W_{g(x)} \subseteq A \;\wedge\; W_{h(x)} \subseteq B.$$

Uniformly for each x, we construct a c.e. set H_x. By the recursion theorem we may assume that we have two computable functions g_x, h_x such that for all n:

$$n \in H_x \;\Leftrightarrow\; W_{g_x(n)} \subseteq A \;\wedge W_{h_x(n)} \subseteq B.$$

If $x \notin K$ then, for all n, we enumerate n into H_x iff $n \in K$. If $x \in K$ we let $H_x = \omega$. Let $M_x = \bigcup_{n \in \omega} W_{g_x(n)}$. Note that $M_x \subseteq A$ if $x \in K$, by the definition of H_x.

Now there are two cases:

(i) If there is an $x \notin K$ such that $M_x \subseteq A$, then $W_{g_x(n)} \subseteq A$ for all n. But this means that $n \in K$ iff $n \in H_x$ iff $W_{h_x(n)} \subseteq B$, i.e., $K \leq_{\mathrm{Q}} B$.

(ii) If for all $x \notin K$ we have $M_x \not\subseteq A$, then for all x,

$$x \in K \Leftrightarrow M_x \subseteq A,$$

i.e., $K \leq_{\mathrm{Q}} A$.

In either case we are done. □

By the Join Theorem for $\mathbf{0}'$ by Posner and Robinson ([23], see also [17, Theorem 5.1] or [22, Theorem XI.3.10]) any noncomputable incomplete T-degree is T-cuppable (even by a low degree). Therefore we will consider in the following only T-cuppability in the c.e. degrees.

Marchenkov showed that every c.e., semirecursive, and η-maximal set is T-incomplete, see [18] or [21, III.5]. Indeed, the proof proceeds by showing that such a set is not Q-complete. By the previous result, it is therefore not Q-cuppable. However, the next result shows that such a set may very well be T-cuppable in the c.e. degrees.

Theorem 7.2 *There is a c.e., semirecursive, and η-maximal set A, for some positive equivalence relation η, that is T-cuppable by a low c.e. set.*

Proof We modify Degtev's construction [2] as presented in [21, III.5.19] and assume that the reader is familiar with it.

In particular we are using the sets B_i^s as explained there to construct the needed positive equivalence relation η. The idea is that in each stage s we have the finite equivalence classes B_i^s, $i \in \omega$, that partition $\overline{A_s}$ such that $\max(B_i^s) < \min(B_{i+1}^s)$ for all i. The i-state of B_i^s is the highest i-state of a member of B_i^s at stage s (i-states of numbers being defined as in the well-known maximal set construction). Now if there exist $e < e' < s$ such that the e-state of $B_{e'}^s$ is higher than the e-state of B_e^s then we may let B_e^s increase its e-state by putting $B_e^{s+1} = \bigcup_{i=e}^{e'} B_i^s$ and $B_{e+j}^{s+1} = B_{e'+j}^s$, for all $j > 0$. If we can arrange that each B_e^s converges to a finite set and can maximize its e-state, then it follows as in the maximal set construction that A is η-maximal with respect to the constructed positive equivalence relation.

In addition to A we construct a c.e. low set L and a Turing reduction Φ^X such that for all x, $K(x) = \Phi^{L \oplus A}(x)$.

Since adding requirements for lowness and satisfying them by finite-injury is standard, the crucial point lies in the definition of Φ:

We define $\mathrm{use}_s(x)$, the use of $\Phi^X(x)$ at the end of stage s as follows:

$$\mathrm{use}_0(x) = \{2x\} \cup \{2y+1 : y \leq x\}$$

$$\mathrm{use}_{s+1}(x) = \begin{cases} \{2x\} \cup \{2y+1 : y \leq \min(B_x^{s+1})\} & \text{if } A \text{ changes at or below} \\ & \quad \max\{y : 2y+1 \in \mathrm{use}_s(x)\} \\ & \quad \text{in stage } s+1; \\ \mathrm{use}_s(x) & \text{otherwise.} \end{cases}$$

Note that the definition is valid since $\min(B_x^s)$ is nondecreasing in s and $B_x^0 = \{x\}$.

For the standard enumeration of Turing reductions $\Phi_i, i \in \omega$, let $u(i,s)$ denote the use of $\Phi_{i,s}^{L_s}(i)$ if it is defined after s steps of computation, and 0 otherwise.

We have to satisfy the following negative requirements:

$$N_{2i}: \quad B_i \text{ is finite}$$

$$N_{2i+1}: \quad \exists^\infty s\ \Phi_{i,s}^{L_s}(i)\downarrow \rightarrow\ \Phi_i^L(i)\downarrow$$

N_{2i} will help to make sure that A is indeed η-maximal, and N_{2i+1} is the usual lowness requirement for L, see [27], Chap. VII, 1. In addition we have the positive requirement that A is semirecursive. We satisfy this by dumping as in the original construction. Finally, we have the positive requirement that $K(x) = \Phi^{L\oplus A}(x)$ for all x. We satisfy this by either enumerating x into L or enumerating

$$m_s(x) = \max\{y \notin A_s : 2y+1 \in \mathrm{use}_s(x)\}$$

into A, if x appears in K at stage $s+1$.

Let $x_t, t \in \omega$, be a computable enumeration without duplication of K.

At stage $s+1$ we do the following:
1. $s = 2t$ even
For $i \leq t$ define restraints as follows:

$$r(2i, s) = \max(B_i^s)$$

$$r(2i+1, s) = u(i, t)$$

Let $q(2i) = m_s(x_t)$ and $q(2i+1) = x_t$. Let $i' \leq t$ be minimal such that $q(i') \leq r(i', s)$. If the minimum does not exist let $i' = t$.

If i' is even, enumerate x_t into L. If i' is odd, choose z such that $m_s(x_t) \in B_z^s$ and dump all B_j^s into A for $j = z, \ldots, \max(z, s)$.

Note that in this way we preserved all requirements N_i, with $i \leq i'$.

2. s odd
Let the sets B_e^s maximize their e-state as explained above.
This completes the construction.

Verification:

Lemma 1: For all i, $\lim_s r(i, s)$ exists and is finite and N_i is satisfied.
Proof: Using induction on i we may assume that $\lim_s r(j, s)$ exists and is finite for all $j < i$. Choose s_0 such that $r(j, s) = r(j, s_0)$ for all $s \geq s_0$ and all $j < i$. Let r_0 denote the maximum restraint (let $r_0 = -1$ for $i = 0$).

All B_j, with $2j < i$, are fixed finite sets by stage s_0. Let m_0 be their maximum element. Note that $m_0 = \max(B_{k_0})$ for the maximum k_0 with $2k_0 < i$. For $i = 0$, let $k_0 = m_0 = -1$.

Let $m' = \min\{m > m_0 : m \notin A_{s_0}\}$, i.e., $m' = \min(B_{k_0+1}^{s_0})$. Note that for all $x \geq m'$ we trivially have $2m' + 1 \in \text{use}_0(x)$.

By induction on s we show that for all s:

$$P_s : \; x \geq m' \;\Rightarrow\; 2\min(B_{k_0+1}^s) + 1 \in \text{use}_s(x).$$

Assume P_s holds, then we show that P_{s+1} holds:

If $B_{k_0+1}^s$ is dumped into A in stage $s+1$, with s even, then A changes at or below $\min(B_{k_0+1}^s)$. Since $x \geq k_0+1$, it follows that $2\min(B_{k_0+1}^{s+1})+1 \in \text{use}_{s+1}(x)$.

If $B_{k_0+1}^s$ is 'eaten' by some B_k^s, $k \leq k_0$, in stage $s+1$, with s odd, then $s \leq s_0$. Thus, $\min(B_{k_0+1}^{s+1}) \leq m'$ and so trivially $2\min(B_{k_0+1}^{s+1})+1 \in \text{use}_{s+1}(x)$.

This completes the inductive step.

Choose $s_1 > s_0$ such that $x_t \geq \max(m', r_0 + 1)$ for all $2t \geq s_1$. For all $s = 2t \geq s_1$ it follows by P_s that $2\min(B_{k_0+1}^s) + 1 \in \text{use}_s(x)$. Since $\min(B_{k_0+1}^s) \notin A_s$ we have $m_s(x_t) \geq m' > m_0$ and so it follows,

$$q(j) > r(j, s) \text{ for all } j < i.$$

This means that requirement N_i is preserved in stage $s+1$ and so we are done:

a.) i even, i.e., $i = 2(k_0 + 1)$. For all stages $s + 1 > s_1$ $B_{k_0+1}^s$ will never be dumped into A nor will $B_{k_0+1}^s$ be eaten by some $B_k^s, k \leq k_0$, since these sets never change again. Thus, the only way it increases is by eating some B_k^s with $k > k_0 + 1$ that has a higher $(k_0 + 1)$-state. But this can happen only finitely often. Therefore $\max_s(B_{k_0+1}^s)$ exists and is finite, i.e., N_i is satisfied.

b.) i odd. Let $i = 2k_0 + 1$. In all stages $s + 1 > s_1$, $s = 2t$, the use of $\Phi^{L_t}_{k_0,t}(k_0)$ is preserved, if the computation converges. It follows that $r(i, s)$ has a finite limit and that N_i is satisfied.

Lemma 2: L is low.
Proof: This follows from Lemma 1 since all lowness requirements are satisfied.

Lemma 3: A is semirecursive and η-maximal.
Proof: This follows from Lemma 1 and the arguments outlined above.

Lemma 4: $K \leq_T L \oplus A$.
Proof: By the construction, we have $\Phi^{L_s \oplus A_s}_s(x_t) = 1$, for all $s > 2t$ and $\Phi^{L_s \oplus A_s}_s(x) = 0$, for all s and $x \notin K$. Since all B_x are finite it follows that the use of Φ is finite and therefore it is a valid Turing reduction from K to $L \oplus A$. □

Remark 7.3 It is straightforward to combine the construction in [21, III.5.19] with permitting. Therefore for any c.e., noncomputable set A there is a c.e., noncomputable, semirecursive, η-maximal set B such that $B \leq_T A$. Since there exist c.e., noncomputable sets that are not T-cuppable in the c.e. degrees (see [27, XIII, 4.4]), it follows that there are also c.e., noncomputable, semirecursive, η-maximal sets that are not T-cuppable in the c.e. degrees.

Acknowledgments

The authors would like to thank Rodney G. Downey and Peter A. Fejer for pointing out related work. Thanks are due to the referee for comments and suggestions to make the paper easier accessible.

Notes

1. For sets, $A \oplus B$ denotes the *join* of the two sets, $\{2a : a \in A\} \cup \{2b+1 : b \in B\}$. The context should be clear in all cases.

2. Parity reducibility is also known as *linear* reducibility.

References

[1] Davis, M., editor, *The Undecidable*, Dover Publications Inc., Mineola, NY, 2004. Basic papers on undecidable propositions, unsolvable problems and computable functions, Corrected reprint of the 1965 original.

[2] Degtev, A. N., "*tt*- and *m*-degrees (Russian)," *Algebra i Logika*, vol. 12 (1973), pp. 143–161. English translation in *Algebra and Logic*, 12:78–89, 1974.

[3] Degtev, A. N., "Some results on upper semilattices and m-degrees (Russian)," *Algebra i Logika*, vol. 18 (1979), pp. 664–679. English translation in *Algebra and Logic*, 18:420–430, 1980.

[4] Dekker, J. C. E., "Infinite series of isols," pp. 77–96 in *Proc. Sympos. Pure Math., Vol. V*, American Mathematical Society, Providence, R.I., 1962.

[5] Dekker, J. C. E., and J. R. Myhill, "Some theorems on classes of recursively enumerable sets," *Trans. Amer. Math. Soc.*, vol. 89 (1958), pp. 25–59.

[6] Downey, R. G., "Two theorems on truth table degrees," *Proc. Amer. Math. Soc.*, vol. 103 (1988), pp. 281–287.

[7] Downey, R. G., "A note on btt-degrees," Technical report, Victoria University, Wellington, New Zealand, September 1998.

[8] Downey, R. G., "A note on btt-degrees," *Rend. Sem. Mat. Univ. Politec. Torino*, vol. 58 (2000), pp. 449–456.

[9] Downey, R. G., and C. G. Jockusch, "T-degrees, jump classes, and strong reducibilities," *Trans. Amer. Math. Soc.*, vol. 301 (1987), pp. 103–136.

[10] Fenner, S., and M. Schaefer, "Bounded immunity and *btt*-reductions," *MLQ Math. Log. Q.*, vol. 45 (1999), pp. 3–21.

[11] Jockusch, C. G., Jr., "Semirecursive sets and positive reducibility," *Trans. Amer. Math. Soc.*, vol. 131 (1968), pp. 420–436.

[12] Kobzev, G. N., "On complete btt-degrees (Russian)," *Algebra i Logika*, vol. 13 (1974), pp. 22–25. English translation in *Algebra and Logic*, 13:10–12, 1974.

[13] Kummer, M., "On the complexity of random strings (extended abstract)," in *13th Annual Symposium on Theoretical Aspects of Computer Science*, volume 1046 of *Lectures Notes in Computer Science*, pp. 25–36, Grenoble, France, 22–24 February 1996. Springer-Verlag, Berlin.

[14] Lachlan, A. H., "A note on universal sets," *J. Symbolic Logic*, vol. 31 (1966), pp. 573–574.

[15] Lachlan, A. H., "On the lattice of recursively enumerable sets," *Trans. Amer. Math. Soc.*, vol. 130 (1968), pp. 1–37.

[16] Lachlan, A. H., "wtt-complete sets are not necessarily tt-complete," *Proc. Amer. Math. Soc.*, vol. 48 (1975), pp. 429–434.

[17] Lerman, M., *Degrees of unsolvability*, Springer-Verlag, Berlin, 1983.

[18] Marchenkov, S. S., "A certain class of incomplete sets (Russian)," *Mat. Zametki*, vol. 20 (1976), pp. 473–478. English translation in *Math. Notes Acad. Sci. USSR*, 20:823–825, 1976.

[19] Muchnik, A. A., and S. Y. Positselsky, "Kolmogorov entropy in the context of computability theory," *Theoret. Comput. Sci.*, vol. 271 (2002), pp. 15–35.

[20] Nies, A., and A. Sorbi, "Structural properties and Σ^0_2 enumeration degrees.," *J. Symbolic Logic*, vol. 65 (2000), pp. 285–292.

[21] Odifreddi, P., *Classical recursion theory*, North-Holland, Amsterdam, 1989.

[22] Odifreddi, P., *Classical recursion theory*, volume II, North-Holland, Amsterdam, 1999.

[23] Posner, D. B., and R. W. Robinson, "Degrees joining to $\mathbf{0}'$," *J. Symbolic Logic*, vol. 46 (1981), pp. 714–722.

[24] Post, E. L., "Recursively enumerable sets of positive integers and their decision problems," *Bull. Amer. Math. Soc.*, vol. 50 (1944), pp. 284–316.

[25] Robinson, R. W., "Simplicity of recursively enumerable sets," *J. Symbolic Logic*, vol. 32 (1967), pp. 162–172.

[26] Schaefer, M., "Simple sets are not btt-cuppable," Technical Report TR-97-09, University of Chicago, September 1997.

[27] Soare, R. I., *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin, 1987.

[28] Soare, R. I., "Computability and recursiveness," *Bull. Symbolic Logic*, vol. 2 (1996), pp. 284–321.

INIT GmbH
Kaeppelestrasse 6,
D-76131 Karlsruhe,
Germany.
mkummer@init-ka.de

DePaul University
CTI, 243 S. Wabash, Ste 401,
Chicago, IL 60604,
USA
mschaefer@cs.depaul.edu