# Strong Reductions and Immunity for Exponential Time

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**Abstract**

This paper investigates the relation between immunity and hardness in exponential time. The idea that these concepts are related originated in computability theory where it led to Post's program. It has been continued successfully in complexity theory [10, 15, 21]. We study three notions of immunity for exponential time. An infinite set $A$ is called

- **EXP**-immune, if it does not contain an infinite subset in **EXP**;
- **EXP**-hyperimmune, if for every infinite sparse set $B \in$ **EXP** and every polynomial $p$ there is an $x \in B$ such that $\{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$ is disjoint from $A$;
- **EXP**-avoiding, if the intersection $A \cap B$ is finite for every sparse set $B \in$ **EXP**.

**EXP**-avoiding sets are always **EXP**-hyperimmune and **EXP**-hyperimmune sets are always **EXP**-immune but not vice versa. We analyze with respect to which polynomial-time reducibilities these sets can be hard for **EXP**. **EXP**-immune sets cannot be conjunctively hard for **EXP** although they can be hard for **EXP** with respect to disjunctive and parity-reducibility. **EXP**-hyperimmunes sets cannot be hard for **EXP** with respect to any of these three reducibilities. There is a relativized world in which there is an **EXP**-avoiding set which is hard with respect to positive truth-table reducibility. Furthermore, in every relativized world there is some **EXP**-avoiding set which is Turing-hard for **EXP**.

**Keywords.** Computational and structural complexity, hardness for exponential time, polynomial time reducibilities.

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# 1 From Post's Program to Complexity Theory

Concepts of immunity have a long tradition in computability theory beginning with the famous paper of Post [12, 19] which introduced simple sets and showed that they are not hard in the sense that the halting problem cannot be many-one reduced to a simple set or its complement. In fact, no set without an infinite computable subset can be many-one hard for the halting problem. These sets are called *immune* and the present paper extends the study of resource bounded versions of this notion. Post also considered the more restrictive notions of hyperimmune and hypersimple sets which are not truth-table hard for the halting problem. Post's program to find a Turing-incomplete set defined by abstract properties more restrictive than being simple was completed in the 1970s when Dëgtev and Marchenkov proved that $\eta$-maximal semirecursive simple sets exist for a suitable positive equivalence-relation $\eta$ and that such sets neither have the Turing-degree of the empty set nor of the halting problem, but are intermediate [18, Section III.5]. Furthermore, Harrington and Soare [13] found a condition which is definable in the lattice defined by set-inclusion of the recursively enumerable sets and which enforces that these sets have an intermediate Turing degree.

Within complexity theory, Berman and Hartmanis [9] started the search for structural properties which imply that a set is not hard for the classes **NP**, **PSPACE** or **EXP**. Berman [8], for example, showed that **NEXP**-complete sets are not **EXP**-immune. This result was strengthened by Tran [22] who showed that **NEXP**-complete sets, and their complements, are not **P**-immune. As a matter of fact, 1-tt and conjunctively complete sets for **NEXP** are even **P**-levelable, and therefore not **P**-immune [21]. Hartmanis, Li and Yesha [15] studied whether **NP**-simple sets can be complete for **NP** where a set $A$ is **NP**-simple iff $A \in \mathbf{NP}$ and $A$ is co-infinite and no infinite set $B \in \mathbf{NP}$ is disjoint to $A$. They showed that an **NP**-simple set cannot be many-one hard, unless every problem in **NP** can be decided in subexponential time. Agrawal communicated to the first author, that under the assumption that **P** and **NP** are not equal, no **NP**-simple set $A$ can be complete for **NP** with respect to honest bounded truth-table reducibility. Schaefer and Fenner [21] showed that no **NP**-hyperimmune set is hard for **NP** with respect to honest Turing reducibility if $\mathbf{P} \neq \mathbf{UP}$, that is, there is a set $A \in \mathbf{NP} \setminus \mathbf{P}$ which has unique witnesses, in the sense that we can write $A = \{x : (\exists y \in \{0,1\}^{p(|x|)})[(x,y) \in B]\}$ where $p$ is a polynomial, $B \in \mathbf{P}$ and there is, for every $x$, at most one $y \in \{0,1\}^{p(|x|)}$ with $(x,y) \in B$. Furthermore, under various stronger assumptions similar results were shown with respect to arbitrary, not necessarily honest polynomial time reducibilities. For example, **NP**-simple sets cannot be bounded truth-table hard for **NP**, unless $\mathbf{SUBEXP} \subseteq \mathbf{UP} \cap \mathbf{coUP}$.

The class **EXP** is much more well-behaved than **NP**. It is different from **P** in all relativized worlds and therefore contains difficult sets. Furthermore, one can build sets in **EXP** by doing a polynomially length-bounded search within other sets in **EXP**. For example, one can construct for any set in **EXP** a sparse but infinite subset which is still in **EXP** (a set is *sparse*, if it contains at most polynomially many strings at every length). The study of immunity notions for **EXP** often yields results like this which are true for all relativized worlds and do not depend on any unproven assumptions such as

the non-collapse of the polynomial hierarchy or some **NP**-complete problem not being computable in subexponential time.

In the present paper, we investigate three immunity-notions for **EXP**. Namely, an infinite set $A$ is

- **EXP**-immune, if it does not contain an infinite subset in **EXP**;
- **EXP**-hyperimmune, if for every infinite sparse set $B \in \mathbf{EXP}$ and every polynomial $p$ there is an $x \in B$ such that $\{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$ is disjoint from $A$;
- **EXP**-avoiding, if the intersection $A \cap B$ is finite for every sparse set $B \in \mathbf{EXP}$.

Every **EXP**-avoiding set is **EXP**-hyperimmune and every **EXP**-hyperimmune set is **EXP**-immune. Note that the condition of $B$ being sparse is necessary in the definition of **EXP**-avoiding sets, since every infinite set has an infinite intersection with a set in **P** namely $\{0,1\}^*$.

In this paper we investigate whether sets immune in one of the senses above can be hard for **EXP** for different types of reducibilities. This continues similar research by Schaefer and Fenner [21] on **NP**-simple, **NEXP**-simple and **NP**-immune sets.

# 2 Basic Definitions and Theorems

The complexity classes **P**, **Q** and **EXP** denote the sets of languages for which there is a constant $c$ such that their characteristic function is computable with time bound $n^c$, $n^{\log^c(n)}$ and $2^{n^c}$, respectively, where $n$ is the length of the corresponding input (viewed as a binary string; actually $n$ has to be set to 3 if the string has length 0, 1 or 2). **NP** is the class corresponding to **P** which permits non-deterministic computations, that is, $A \in \mathbf{NP}$ iff there is a function $M$ and a constant $c$ such that, for all $x \in A$, some computation of $M$ with input $x$ halts in time $n^c$ and, for all $x \notin A$, no computation with input $x$ halts, whatever time the computation needs. Note that non-determinism permits $M$ to have different computations on the same input, this is an essential part of the definition of **NP**. A set is *sparse*, if for some polynomial $p$ we have $|A \cap \{0,1\}^n| \leq p(n)$ for all $n$.

The following definition of an **EXP**-immune set is analogous to those of **P**-immune, **NP**-immune and **PSPACE**-immune sets found in the literature [6].

**Definition 2.1.** *A set $A$ is* **EXP***-immune iff $A$ is infinite and it does not contain an infinite subset in* **EXP***.*

We observed earlier that every infinite set in **EXP** has an infinite sparse subset in **EXP**, it follows that a set $A$ is **EXP**-immune iff $A$ does not have an infinite sparse subset in **EXP**. This property is strengthened in the following definition of **EXP**-avoiding sets.

**Definition 2.2.** *A set $A$ is* **EXP***-avoiding iff $A$ is infinite and the intersection of $A$ with any sparse set in* **EXP** *is finite.*

The next definition is obtained by adapting the notion of **NP**-hyperimmune from Schaefer and Fenner [21] to exponential time. In the definition we use the notation "$f\{x\}$" to

indicate that the output of $f$ is not a string but a set of strings which, of course, could be coded as a string again.

**Definition 2.3.** *Call a (partial) function $f$ an* **EXP**-array, *if $f$ is computable in* **EXP**, *it has infinite domain and there is a polynomial $p$ such that the cardinality of $f\{x\}$ is at most $p(|x|)$ for all $x$ and $p^{-1}(|x|) \leq |y| \leq p(|x|)$ for all $y \in f\{x\}$, where $p^{-1}(n) = \min\{m : p(m) \geq n\}$. (The final condition assures that $f$ is honest with respect to every element it outputs.)*

*A set $A$ is* **EXP**-*hyperimmune iff for all* **EXP**-*arrays $f$ there is an $x$ in the domain of $f$ such that $f\{x\}$ and $A$ are disjoint.*

The following theorem shows that the definition of **EXP**-hyperimmunity given in Definition 2.3 is equivalent to the one given in the introduction.

**Theorem 2.4.** *A set $A$ is* **EXP**-*hyperimmune iff for every infinite* **EXP**-*sparse set $B$ and every polynomial $p$, there is an $x \in B$ such that $\{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$ is disjoint from $A$.*

**Proof.** Assume that $A$ is **EXP**-hyperimmune, $B$ is an infinite **EXP**-sparse set and $p$ is an increasing polynomial with $p(n) > n$ for all $n$. Then define the function $f$ by taking $B$ as the domain of $f$ and

$$f\{x\} = \{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$$

Clearly, $f$ is computable in exponential time. As $B$ is sparse, there is an increasing polynomial $q$ such that $B \cap \{0,1\}^n$ contains less than $q(n)$ elements for every $n$. It follows that $f\{x\}$ has at most $q(p(|x|)) \cdot p(|x|)$ many elements, and so $f$ is an **EXP**-array. Hence, there is an $x$ such that $f\{x\}$ is disjoint from $A$ and thus the condition given in the theorem is satisfied by every **EXP**-hyperimmune set.

For the converse direction, consider any set $A$ satisfying the condition of the theorem and let $f$ be an **EXP**-array with respect to an increasing polynomial $q$ where $q(n) > n$ for all $n$. Let $p(n) = q(q(n)) + 1$ and define a set $B = B_0 \cup B_1 \cup \ldots$ in stages as follows.

**Stage** 0**:** Let $B_0 = \emptyset$ and $l_0 = 0$.

**Stage** $s+1$**:** Check whether there is a string $x$ in the domain of $f$ such that $l_s \leq |x| \leq p(l_s)$ and all elements $y \in f\{x\}$ have at least length $l_s$.

(I) If so, let $B_{s+1} = B_s \cup f\{x\}$ for the first such $x$ found and let $l_{s+1} = p(|x|) + 1$.

(II) If not, let $B_{s+1} = B_s$ and $l_{s+1} = l_s + 1$.

For the verification, one sees easily that the set $B = \cup_s B_s$ has at length $n$ at most $p(p(n))$ many elements and is thus sparse. Let $r$ be a polynomial such that $f$ is computable in time $2^{r(|x|)}$ for all $x$ in its domain. Then one can do any step $s$ with $l_s \leq n$ in time $O(2^{r(p(n))} \cdot 2^{p(n)})$ and thus compute the characteristic function of $B$ in time $O(n \cdot 2^{r(p(n))} \cdot 2^{p(n)})$. So $B$ is in **EXP**.

Now there is a $y \in B$ such that $B \cap A$ does not contain any element of length $n$ with $p^{-1}(|y|) \leq n \leq p(|y|)$. This $y$ came into $B$ as an element of some $f\{x\}$ and therefore

$q^{-1}(|x|) \leq |y| \leq q(|x|)$. This condition is equivalent to $q^{-1}(|y|) \leq |x| \leq q(|y|)$. As $p(m) > q(q(m))$ for all $m$, it follows that every element of $f\{x\}$ has a length $n$ with $p^{-1}(|y|) \leq n \leq p(|y|)$ and thus $f\{x\}$ is disjoint to $A$. So $A$ is **EXP**-hyperimmune. □

If the intersection of $A$ with an infinite sparse set $B$ is finite, then the condition in the above theorem is clearly satisfied for every polynomial. Thus one obtains the following corollary.

**Corollary 2.5.** *Every* **EXP***-avoiding set is* **EXP***-hyperimmune.*

A reducibility is an algorithm to compute a set $A$ relative to a set $B$. $B$ is often called an *oracle*. In this paper we only consider polynomial time algorithms for reducibilities. Different types of reducibilities are obtained by restricting the algorithm, and its access to $B$; we include a partial list.

**Turing reducibility.** $A$ is Turing reducible to $B$ iff there is a polynomial $p$ such that some Turing machine $M$ computes $A(x)$ in time $p(|x|)$ with queries to $B$. We write $A(x) = M^B(x)$. Due to time constraints there are at most $p(|x|)$ queries and every $y$ in a query satisfies $|y| \leq p(|x|)$. If it also satisfies $|y| \geq p^{-1}(|x|)$ for every $y$ queried by $M$, then the reduction is called *honest*.

**Truth-table reducibility (tt-reducibility).** $A$ is tt-reducible to $B$ iff there is a polynomial-time Turing machine $M$ and a polynomial-time computable function $f$ such that $f\{x\}$ is a set of polynomially many strings and $M$ computes $A$ relative to $B$ only querying elements of the set $f\{x\}$.

**Positive reducibility (ptt-reducibility).** $A$ is ptt-reducible to $B$ iff $A$ is tt-reducible to $B$ via a function $f$ and machine $M$ as defined previously and, moreover, for any two sets $D \subseteq E$ and any input $x$ we have that $M^D(x) \leq M^E(x) \in \{0, 1\}$.

**Conjunctive reducibility (c-reducibility).** $A$ is c-reducible to $B$ iff there is a polynomial time computable function $f$ such that $x \in A$ iff $f\{x\} \subseteq B$.

**Disjunctive reducibility (d-reducibility).** $A$ is d-reducible to $B$ iff there is a polynomial time computable function $f$ such that $x \in A$ iff $f\{x\}$ meets $B$, that is, $f\{x\} \cap B \neq \emptyset$.

**Parity-reducibility.** $A$ is parity-reducible to $B$ iff there is a polynomial time computable function $f$ such that $x \in A$ iff the cardinality of the intersection $f\{x\} \cap B$ is odd. In computability theory, parity-reducibility is often called linear reducibility [18, page 269].

**Many-one reducibility (m-reducibility).** $A$ is m-reducible to $B$ iff there is a polynomial time computable function $f$ such that $x \in A$ iff $f(x) \in B$.

The function $f$ in the definitions of positive, conjunctive, disjunctive and parity-reducibility has as output not a single string but a set of strings. This is indicated by writing $f\{x\}$ instead of $f(x)$. Furthermore, a $g(n)$-r-reducibility is a reducibility where on input of length $n$ one can ask at most $g(n)$ questions. For example, a $\log(n)$-tt-reduction

requires that the cardinality of the set $f\{x\}$ is always at most $\log(|x|)$.

In the following, reducibilities are also used to define a generalization of the notion of classes using advice. The best-known related concept is the class **P/poly** which is the class of all sets that can be Turing-reduced to polynomial-sized advice.

**Definition 2.6.** *We call a set $E$ in* **EXP** compressible via r-reducibility*, if there is an r-reduction $M$ and for infinitely many lengths $n > 0$ there is a set $A_n \subseteq \{0,1\}^{<n}$, called the* advice *such that $M$ r-reduces $E$ to $A_n$ on the domain $\{0,1\}^n$. Otherwise $E$ is called* incompressible via r-reducibility.

*We say* **EXP** *is* compressible via r-reducibility *iff every set $E$ in* **EXP** *is compressible via r-reducibility, and* incompressible via r-reducibility*, otherwise.*

**Theorem 2.7.** **EXP** *is incompressible via any of the following reducibilities: conjunctive reducibility, disjunctive reducibility, parity-reducibility. This result relativizes.*

Note that **EXP** $\subseteq$ **P/poly** iff there is a tally Turing-complete set for **EXP**. Wilson [24] constructed a relativized world in which **EXP** $\subseteq$ **P/poly**.

Hence, we cannot expect to improve the statement of the theorem to Turing reductions without making further assumptions.

**Proof.** Recall that the reducibilities above compute on input $x$ some set $f\{x\}$ such that $x \in E$ iff $f\{x\}$ intersects $A_n$ in case of disjunctive reducibility, $f\{x\}$ is a subset of $A_n$ in the case of conjunctive reducibility, and $f\{x\}$ has an odd number of elements in common with $A_n$ in the case of parity-reducibility.

**Disjunctive Reducibility.** Given a reducibility $f = \varphi_e$ which — without loss of generality — is computable in time $2^e$ on input of length $e$ one defines a partial function $g$ from $\{0,1\}^e$ to $\{0,1\}^{<e}$ which on input $x$ outputs some string $z$ iff this $z$ (and perhaps some other ones) is in the set $f\{x\}$ but not in any set $f\{y\}$ with $y \in \{0,1\}^e \setminus \{x\}$. Therefore, whenever $g(x)$ and $g(y)$ are defined, then $g(x) \neq g(y)$. As there are $2^e$ strings of length $e$ but only $2^e - 1$ strings of strictly shorter length, there is a string $x \in \{0,1\}^e$ such that $g(x)$ is undefined. For each $e$, let $x_e$ be the lexicographically first string in $\{0,1\}^e$ where $g$ is undefined and let $E = \{x_0, x_1, \ldots\}$ be the set of all these $x_e$.

To see that $E$ is in **EXP** note that $g$ is based on the function $f = \varphi_e$ which is computable in time $2^e$. We initialize an array of length $2^e$ with entry 0 for every $x \in \{0,1\}^e$. For every $z$ we can check in time $4^e$ whether $z \in \varphi_e\{x\}$ for exactly one $x \in \{0,1\}^e$ and if so, set the corresponding entry to 1. Repeating this for all $z$ allows us to compute $x_e$ as the lexicographic first string in $\{0,1\}^e$ whose entry is still 0 in at most $8^e$ steps.

For every $e$, the function $f = \varphi_e$ does not compute $E \cap \{0,1\}^e$ using any advice $A_e$, because $x_e \in E$ implies that $A_e$ intersects $\varphi_e\{x_e\}$ at some element $z$ and, since $g(x_e)$ is undefined, there is some $y \in \{0,1\}^e \setminus \{x_e\}$ with $z \in \varphi_e\{y\}$. It would follow that $y \in E$ which contradicts $x_e$ being the only element of $E$ of length $e$.

**Conjunctive Reducibility.** The proof is analogous to the one for disjunctive reducibility. We can take the complement $\overline{E}$ of the previously constructed set $E$ and use the fact that $\varphi_e$ reduces $E$ disjunctively to advice $A_e$ iff $\varphi_e$ reduces $\overline{E}$ conjunctively to $\overline{A_e}$.

**Parity-Reducibility.** For the case of parity-reducibility we use the fact that every $f = \varphi_e$ defines a linear mapping from the $(2^n - 1)$-dimensional Boolean vector space

of the subsets of $\{0,1\}^{<n}$ into the vector space of the subsets of $\{0,1\}^n$ for each $n$. As the space of all characteristic functions on $\{0,1\}^n$ is $2^n$-dimensional, there is some possible characteristic function on $\{0,1\}^n$ which is not in the linear closure of the images $\varphi_e^{-1}(\{z\})$ where $|z| < n$. Hence, for every length $n = e$, we can determine in exponential time the characteristic function on all strings of length $e$ in such a way that it does not coincide with any possible image $\varphi_e^{-1}(A_e)$ for any $A_e \subseteq \{0,1\}^{<n}$. Thus we get a set $E \in \mathbf{EXP}$ which is incompressible via parity-reducibility. □

# 3 Immunity and Hardness for EXP

The goal of this section is to investigate for which reducibilities $r$ there are **EXP**-immune, **EXP**-hyperimmune, and **EXP**-avoiding sets which are r-hard for **EXP**.

**Theorem 3.1.** *No c-hard set for* **EXP** *is* **EXP***-immune.*

**Proof.** From Theorem 2.7 it follows that there is a set $\{x_0, x_1, \ldots\} \in \mathbf{EXP}$ whose complement $E$ is incompressible via conjunctive reducibility. Assume by way of contradiction that there is a conjunctive reduction $f$ from $E$ to some **EXP**-immune set $A$ and let $F = \{x : f\{x\} \text{ contains some string } z \text{ with } |z| \geq |x|\}$. If $E \cap F$ is infinite then the set

$$U = \{z : (\exists x \in E \cap F)\,[|x| \leq |z| \wedge z \in f\{x\}]\}$$

is also infinite and it is a subset of $A$, since $f\{x\} \subseteq A$ for every $x \in E$. This contradicts $A$ being **EXP**-immune. So $E \cap F$ is finite and we can modify $f$ to obtain the following function $g$ also computable in polynomial time. Without loss of generality let $\lambda \in A$. Now $g$ is defined by taking the first of the following cases which applies.

$$g\{x\} = \begin{cases} f\{x\} & \text{if } f\{x\} \subseteq \{0,1\}^{<|x|}; \\ \{\lambda\} & \text{if } x \in E \cap F; \\ \emptyset & \text{otherwise.} \end{cases}$$

This $g$ would then witness that $E$ can be compressed using a conjunctive reduction, contradicting the choice of $E$. Thus $E$ cannot be c-reduced to an **EXP**-immune set. □

Buhrman [10] showed that there are d-complete sets in **EXP** which are **P**-immune. This result also holds for higher levels.

**Fact 3.2 (Buhrman [10]).** *Let* **DEXP** *denote the class of all sets which are computable in double exponential time, that is, in time* $2^{2^{p(n)}}$ *for some polynomial p. For a given* **DEXP***-complete set A one can construct an* **EXP***-immune set* $B \in \mathbf{DEXP}$ *such that* $x \in A \Leftrightarrow x0 \in B \vee x1 \in B$. *B is clearly d-complete for* **DEXP** *and d-hard for* **EXP**.

In contrast to the existence of **EXP**-immune sets which are d-hard for **EXP**, the next result shows that d-hard sets for **EXP** cannot be **EXP**-hyperimmune.

**Theorem 3.3.** *No d-hard set for* **EXP** *is* **EXP***-hyperimmune.*

**Proof.** Let $E = \{x_0, x_1, \ldots\}$ be the language constructed in Theorem 2.7 which is incompressible with regard to disjunctive reductions. Assume that $E \leq_d A$ for an

hyperimmune set $A$ via a reduction $g$, that is, $x \in E$ iff $g\{x\} \cap A \neq \emptyset$. By the padding-lemma there is an infinite polynomial-time computable set $U$ of indices of $g$. For every $e \in U$ we can compute for every $z \in f_e\{x_e\}$ with $|z| < |x_e|$ one input $y \in \{0,1\}^e \setminus \{x_e\}$ such that $z \in f_e\{y\}$ (this is possible by the definition of $x_e$).

This gives a polynomial-sized subset $F_e$ of $\{0,1\}^e$. We define

$$h\{x_e\} = \{z : (\exists y \in F_e \cup \{x_e\})\,[|z| \geq e \wedge z \in f_e\{y\}]\}$$

where $h\{y\}$ is undefined whenever $y \notin E$ or $|y| \notin U$. As $U$ is infinite, $h$ has an infinite domain and there is an $e$ such that $h\{x_e\} \cap A = \emptyset$. Now $f_e$ disjunctively reduces the elements of $F_e \cup \{x_e\}$ to $A^{<e}$, which contradicts the construction in Theorem 2.7. □

Theorem 3.3 leads to the question whether **EXP**-hyperimmune sets can be parity-hard for **EXP**. The answer is that they cannot and the proof is based on an idea which works for all reducibilities where the membership of $x$ in the first set depends only on the cardinality of $f\{x\}$ intersected the second set where $f$ is a polynomial time computable function representing the reduction. So the proof of the next result could also be used as an alternative proof for Theorem 3.3 or to get the result for modulo-$c$-reductions where $x$ is in the first set iff the intersection of $f\{x\}$ with the second set has a cardinality not being a multiple of $c$.

**Theorem 3.4.** *No* **EXP***-hyperimmune set is hard for* **EXP** *with respect to parity-reducibility.*

**Proof.** Using a priority construction we construct a set $E$ in **EXP** which is not tt-reducible to any **EXP**-hyperimmune set $A$. Let $f_0, f_1, \ldots$ be a list of all polynomial time tt-reducibilities where $f_i$ asks at most $n^i$ many queries and needs at most $2^{n+i}$ many computation steps on inputs of length $n$. By padding arguments it is guaranteed that all polynomial time reductions are covered in this list.

The requirement $R_e$ for $E$ with $e = \langle i, j, k\rangle$ is the following: for every $i, j$ there are $m_e, x_e, y_e$ such that

- $x_e$ is strictly before $y_e$ with respect to length-lexicographic order,
- $j \leq m_e < |x_e| \leq |y_e| < (m_e)^k$,
- $f_i\{x_e\} \cap \{0,1\}^{<m_e} = f_i\{y_e\} \cap \{0,1\}^{<m_e}$,
- $y_e \in E \Leftrightarrow x_e \notin E$.

The $j$ is just a padding parameter to make sure that each requirement is not only satisfied once, but infinitely often. One says that requirement $R_e$ requires attention at $s$ if one can choose $x_e, y_e, m_e$ such that $y_e = s$ and the first three of the four conditions for $R_e$ are satisfied.

We construct $E$ inductively by considering all requirements in length-lexicographic order. If $E$ is already defined for all strings length-lexicographically smaller than $s$ and there is some $R_e$ with $e \leq |s|$ requiring attention but not yet satisfied, then one selects the least such $e$ and defines $s \in E$ iff $x_e \notin E$ for the corresponding $x_e$; note that $y_e = s$.

Otherwise no unsatisfied $R_e$ with $e \leq |s|$ requires attention and one defines $s \in E$ choosing some default option.

First we verify that for every $i$ there is a $k$ such that $\langle i,j,k\rangle$ is satisfied for all $j$. To see this, choose $k = i+2$ and fix $j$ as $j$ is just a padding parameter. Given any $m$, there are at most $(2^m)^{m^i} = 2^{m^{i+1}}$ many different sets of queries of length shorter than $m$. As $2^{m^{k+2}} - 2^m > 2^{m^{i+1}}$ for all sufficiently large $m$, the parameters $x_e, y_e, m_e$ exist whenever $s$ is sufficiently large. So $R_e$ almost always requires attention and is eventually satisfied.

Secondly we verify that $E$ is not parity-reducible to any **EXP**-hyperimmune set $A$. Assume by way of contradiction that $f_i$ were such a reduction, that is, $x \in E$ iff the cardinality of $f_i\{x\} \cap A$ is odd. Now one can construct inductively a set $B$ in stages $m$ starting with $B = \emptyset$.

- Check whether there are $j, x, y$ such that the four conditions of $R_e$ are satisfied with parameters $e = \langle i,j,k\rangle$, $m_e = m$, $x_e = x$ and $y_e = y$.
- If so, update $B$ by putting the set $f_i\{x_e\} \cup f_i\{y_e\} \setminus \{0,1\}^{<m}$ into $B$.
- If not, let $B$ unchanged.

The set $B$ is sparse as every $z \in B$ goes into $B$ at some stage $m$ with $|z| \leq m$ and at most $2m^i$ many strings go into $B$ at stage $m$; in particular, $B$ has at most $2|z|^{i+1}$ strings of length $|z|$. The set $B$ is in **EXP** as the search for the $x_e, y_e$ only considers candidates $x_e, y_e$ of length up to $m^k$. The set $B$ is infinite as for all $j$ the requirement $R_{\langle i,j,k\rangle}$ had been satisfied and the $m_e$ of the requirement $R_e$ is at least $j$ and the search does not need the original $x_e, y_e$ but works with any $x_{e'}, y_{e'}$ instead which might originate from some triple $e' = \langle i,j',k\rangle$ with $j' > j$. Taking the polynomial $p$ as $p(n) = n^k$, there is an $z \in B$ such that $A \cap B$ does not contain strings having a length between $p^{-1}(|z|)$ and $p(|z|)$ (by Theorem 2.4). It follows that the $z$ belongs to some $m$ in the construction with $m \leq |z| \leq m^k$. Let $x_e, y_e$ be the strings considered at stage $m$. Every string $u$ in the symmetric difference of the queried sets $f_i\{x_e\}, f_i\{y_e\}$ has at least length $m$ and is in $B$. The string $u$ satisfies $p^{-1}(|z|) \leq |u| \leq p(|z|)$ and $u \notin A$. It follows that $f_i\{x_e\} \cap A$ and $f_i\{y_e\} \cap A$ have the same cardinality but exactly one of the strings $x_e, y_e$ is in $E$. Thus, $f_i$ is not a parity-reduction from $E$ to $A$ and $A$ is not hard for **EXP** with respect to parity-reducibility. □

The corresponding question for general truth-table reductions is open. Buhrman [11] proved the related result that no **NEXP**-simple set is btt-hard for **EXP** where a **NEXP**-simple set is a co-infinite set $A \in$ **NEXP** such that no infinite set in **NEXP** is disjoint to $A$. Later, Schaefer [20] showed that no $\alpha \log(n)$-tt-hard set for **EXP** is **EXP**-hyperimmune, where $\alpha$ can be any constant; the proof of this result implied Buhrman's result. It also showed that **NEXP**-simple sets cannot be btt-hard for **EXP**. Buhrman actually showed a stronger result for tt-reductions that comes closer to the general case.

**Fact 3.5 (Buhrman [11]).** **EXP**-*hyperimmune sets are not* $n^\alpha$-*tt-hard for* **EXP** *where* $\alpha < \frac{1}{3}$.

This also implies that **EXP**-hyperimmune sets are not $\alpha \log n$-Turing hard for **EXP** where $\alpha < \frac{1}{3}$. This bound is almost optimal as the next result shows that there is even an **EXP**-avoiding set which is $\log^2(n)$-Turing-hard for **EXP**.

**Theorem 3.6.** *There is an* **EXP***-avoiding set which is* $\log^2(n)$*-Turing-hard for* **EXP***.*

**Proof.** Let $E$ be an many-one-complete set for **EXP** and let $\{y_0, y_1, \ldots\}$ be a set of strings such that $y_n$ has length $n$ and has maximal Kolmogorov complexity with respect to the time-bound $2^{2^n}$. Let $F$ be the set with the characteristic function $y_0 y_1 y_2 \ldots$ with respect to length-lexicographic ordering.

The function $f$ mapping $n$ to the first integer number larger than $\frac{1}{2} \cdot \log^2(n)$ is polynomial-time computable and increasing. Now consider $A = \{xy_{f(|x|)} : x \in E \oplus F\}$. The Kolmogorov complexity with respect to the time bound $2^{2^{f(|x|)}}$ of every string $xy_{f(|x|)}$ is at least $f(|x|) - c$ where $c$ is some constant independent of $|x|$. It follows that the intersection of $A$ and any given sparse set in **EXP** is finite. On the other hand, $A$ is infinite, and so $A$ is **EXP**-avoiding and **EXP**-hyperimmune.

A Turing-reduction from $E \oplus F$ to $A$ first computes for input $x$ the value $f(|x|)$ and then reconstructs $y_{f(|x|)}$ with queries to places $x'y_{f(|x'|)}$ where $x'$ are the positions in the $F$-part of the join $E \oplus F$ coding the bits of $y_{f(|x|)}$. The bits of the $y_{f(|x'|)}$ itself have to be reconstructed first by going down a further iteration and to get these, one has even to get one more step down. But then one can compute these very short strings as every string $y_n$ is computable in time $2^{2^n+n}$. It is easy to see that the number of queries needed by the whole procedure is in $f(|x|) + O(\log(|x|))$. One remaining query to establish whether $x \in E \oplus F$ is added by querying whether $xy_{f(|x|)}$ is in $A$. □

It is unknown what happens in the case of a $\log(n)$-Turing reduction, or a general truth-table reduction. The next result shows that there is at least a relativized world where there is a truth-table hard set for **EXP** which is **EXP**-avoiding. Remember that there is a relativized world in which $\mathbf{EXP} \subseteq \mathbf{P/poly}$ [24].

**Theorem 3.7.** *In the relativized world where* $\mathbf{EXP} \subseteq \mathbf{P/poly}$*, some* **EXP***-avoiding set is hard for* **EXP** *with respect to positive truth-table reducibility.*

**Proof.** If $\mathbf{EXP} \subseteq \mathbf{P/poly}$ then there is a tally Turing-hard set $E$ for **EXP**. Assuming that $0^{2n} \in E \Leftrightarrow 0^{2n+1} \notin E$ and using that only strings in $\{0\}^*$ have to be queried by the Turing reduction, one obtains a set $E$ that is hard for **EXP** with respect to positive truth-table reducibility.

Now let $\{y_0, y_1, \ldots\}$ be a set of strings such that $y_n$ has length $n$ and has maximal Kolmogorov complexity with respect to the time-bound $2^{2^n}$. The set $A = \{a_n y_{|a_n|} : 0^n \in E\}$ is ptt-hard for **EXP** since $|a_n| \leq \log(n) + 2$ for all $n$ and the equation

$$0^n \in E \Leftrightarrow (\exists y \in \{0,1\}^{|a_n|})\,[xy \in A]$$

needs at most $4n$ disjunctive queries.

Every string in $A$ is of the form $xy_{|x|}$ for some $x$. As $y_{|x|}$ has at least Kolmogorov complexity $|x|$ with respect to the time bound $2^{2^{|x|}}$, it follows that the intersection of $A$ and any given sparse set in **EXP** is finite. On the other hand, $A$ is infinite and therefore **EXP**-avoiding and **EXP**-hyperimmune. □

**Corollary 3.8.** *There is a relativized world in which some* **EXP***-hyperimmune set is hard for* **EXP** *with respect to positive truth-table reducibility.*

# 4 Immunity and Related Concepts

In this section, we investigate to which extent immunity notions are compatible with other well-known complexity theoretic properties such as randomness, genericity, approximability, and simplicity.

## 4.1 Randomness and Genericity

Within this section, the notions of **EXP**-Immunity are compared to the notions of resource-bounded randomness and genericity. These notions are effective variants of concepts from measure theory and Baire category. These notions need to be more restrictive than their classical counterparts in order to be meaningful on complexity classes as these classes are countable (except in the case of non-uniform classes like **P/poly**). In a classical sense countable classes are always small; that is, they have measure 0, and are meager. Lutz [17] introduced the following notion of measure 0 and random sets with respect to quasi-polynomial time computations. Quasi-polynomial means time $n^{\log^c(n)}$ for some constant $c$. This class corresponds to **EXP** if one uses inputs of exponential size (as is done in the case of functionals).

**Definition 4.1 (Lutz [17]).** *Let $a_x$ be the $x$-th string with respect to the length-lexicographically ordered list $\lambda, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$ of all strings. That is, $a_0 = \lambda$, $a_1 = 0$, $a_2 = 1$, $a_3 = 00$, and so on. Furthermore, call $f$ a* **Q**-functional *iff the domain of $f$ is the set of prefixes of characteristic functions, and, for arbitrary sets $A$, the value $f(A(a_0)A(a_1)\ldots A_{a_x})$ is computed in quasi-polynomial time, that is, in time $x^{\log^c(x)}$ for some constant $c$.*

*A* **Q**-*functional $f$ is a* **Q**-martingale *iff the values of $f$ are (codes for) positive rational numbers and $f$ satisfies,*

$$f(B(a_0)\ldots B(a_x)) \geq \tfrac{1}{2} \cdot (f(B(a_0)\ldots B(a_x)0) + f(B(a_0)\ldots B(a_x)1)),$$

*for every set $B$, and every $x$. A* **Q**-*martingale $f$* succeeds *on a set $A$ iff for every rational number $r$, there is an $x$ such that $f(A(a_0)A(a_1)\ldots A(a_x)) > r$.*

*A class has* **Q**-measure 0 *iff there is a* **Q**-*martingale which succeeds on every set in the class. A set $A$ is called* **Q**-random *iff no* **Q**-*martingale succeeds on the set $A$.*

Note that a class may fail to have **Q**-measure 0 although it does not contain **Q**-random sets. The most prominent example for such a class is the class **EXP** itself as on the one hand **EXP** does not have **Q**-measure 0 while on the other hand no set in **EXP** is **Q**-random [4, 16, 17]. The next proposition shows that **Q**-random sets cannot be **EXP**-hyperimmune. The proof uses a standard technique which can also be applied to show that general generic sets (as defined below) are not random [1].

**Proposition 4.2.** *The class of all* **EXP**-*hyperimmune sets has* **Q**-*measure* 0*. In particular, no* **EXP**-*hyperimmune set is* **Q**-*random.*

**Proof.** Let $A$ be **EXP**-hyperimmune and let $g$ be the function which maps any string of length $n$ to the set $I_n = \{0^i1^j : i, j \geq 0 \wedge i + j = n\}$. The function $g$ is honest and

total. Thus there are infinitely many sets $I_n$ disjoint to $A$. Now consider the following p-martingales $f_n$ defined by $f_n(\lambda) = 1$ and

$$f_n(b_0b_1\ldots b_xb_{x+1}) = \begin{cases} 1.0 \cdot f_n(b_0b_1\ldots b_x) & \text{if } x \notin I_n; \\ 1.5 \cdot f_n(b_0b_1\ldots b_x) & \text{if } x \in I_n \text{ and } b_{x+1} = 0; \\ 0.5 \cdot f_n(b_0b_1\ldots b_x) & \text{if } x \in I_n \text{ and } b_{x+1} = 1. \end{cases}$$

If $I_n \subseteq \overline{A}$ then $f_n$ converges, on $A$, to the value $1.5^{n+1}$. The sum over all $1.5^{-n-1}$ for $n = 0, 1, \ldots$ is 2. Now let $f$ be the p-functional given by the infinite sum $0.5 \cdot (1.5^{-1} \cdot f_0 + 1.5^{-2} \cdot f_1 + 1.5^{-3} \cdot f_2 + 1.5^{-4} \cdot f_3 + \ldots)$, then each part $1.5^{-n-1} \cdot f_n$ is positive on every set and for those infinitely many $n$ where $A$ is disjoint to $I_n$ the functional $1.5^{-n-1} \cdot f_n$ goes in the limit to 1. It follows that $f$ diverges to $\infty$ on every **EXP**-hyperimmune $A$, and thereby witnesses that the class of all **EXP**-hyperimmune sets has **Q**-measure 0. □

Ambos-Spies, Fleischhack and Huwig [2] introduced a notion of genericity which is compatible with the notion of randomness in the sense that every random set is generic but not vice versa. Lutz [16] transferred the original general definition from Computability theory to complexity theory.

**Definition 4.3 (Ambos-Spies, Fleischhack and Huwig [2]; Lutz [16]).** *A set $A$ is* **Q**-generic *iff for every* **Q***-functional $f$,*

- *either $f(A(a_0)A(a_1)\ldots A(a_x)) \notin \{0,1\}$ for almost all $x$*
- *or $f(A(a_0)A(a_1)\ldots A(a_x)) = A(a_{x+1})$ for infinitely many $x$.*

*The condition $f(A(a_0)A(a_1)\ldots A(a_x)) \notin \{0,1\}$ permits $f$ not to make a prediction. If $f$ makes infinitely many predictions on $A$, then the second case must pertain.*

*A set $A$ is* general **Q**-generic*, if the functional is either almost always undefined, or it infinitely often predicts the next quasi-polynomially many values and one of these predictions is met by $A$. Predicting quasi-polynomially many values means that $f$ predicts $A(a_{x+1})$ up to $A(a_{q(x)})$ where $q(x) = 2^{\log^c(x)}$ for some constant $c$.*

On the one hand, Ambos-Spies [1] showed that no general **Q**-generic set is **Q**-random. On the other hand, every **Q**-random set is still **Q**-generic [3, 4] so that these two notions of genericity are different. It follows from the definition that every **Q**-generic set is **EXP**-hyperimmune. Since any **EXP**-avoiding set $A$ contains only finitely many strings from the set $\{0\}^*$, one can easily show that $A$ is not **Q**-generic by considering a function which predicts every element of $\{0\}^* - A$ to be in $A$: none of these predictions is correct.

**Fact 4.4.** *Every general* **Q***-generic set is* **EXP***-hyperimmune. Therefore, no general* **Q***-generic set is hard for* **EXP** *with respect to d-reducibility, c-reducibility or parity-reducibility. Some but not all* **Q***-generic sets are* **EXP***-hyperimmune. No* **EXP***-avoiding set is* **Q***-generic.*

Theorems 4.5 and 4.6 prove that in the case of truth-table and Turing reducibilities, it depends on the relativized world whether general **Q**-generic sets can be hard for **EXP** or not. Actually, Theorem 4.5 even holds for positive truth-table reducibility in place of truth-table reducibility.

**Theorem 4.5.** *In every relativized world where* $\mathbf{EXP} \subseteq \mathbf{P/poly}$ *there is a truth-table hard set for* **EXP** *which is general* **Q***-generic.*

**Proof.** Since $\mathbf{EXP} \subseteq \mathbf{P/poly}$ there is a set $A$ such that $H = \{0^x : x \in A\}$ is **EXP**-complete. Consider the set $B = \{a_x 10^y : x \in A\}$. There is an enumeration $f_0, f_1, \ldots$ of **Q**-functionals such that every $f_e$ on data of the form $L(a_0)L(a_1)\ldots L(a_x)$ with $n = |a_x| \geq 8$ only predicts $L$ at places of length $n, n+1, \ldots, n^e$. We will now show how to modify $B$ to obtain a set $C$ which is general **Q**-generic. For the construction, let

$$U_n = \{n^{\log^k(n)} : k < \log\log(n)\}$$

for all $n \geq 8$, where $\log(m)$ is the smallest natural number $o$ with $2^o \geq m$. Note that (a) for almost all $n$ and all $k < \log\log(n)$ it holds that $n^{\log^k(n)} < 2^{n-10}$ and that (b) for all $n$ the bound $e < \log(n)$ is satisfied. Moreover, one chooses the $l_e$ to grow so fast that for every $e$ and $n$ with $n \geq 8$ and $l_e \leq \max(U_n)$ it holds that $2^{e+3} < |U_n|$.

**Before stage** 0**:** Let $B_0 = B$. Initialize all restraints $r_e$ to $l_e$ (they might be increased later).

**Stage** $s$**:** Let $B_s$ be the current variant of $B$ before entering stage $s$. Find the smallest $e \leq s$ such that $f_e$ is a **Q**-functional satisfying the following two conditions:

- $f_e$ on input from $B_s$ does not make a correct prediction affecting only values below $r_e$,
- there are $x$ and $m$ such that $m = |a_x|$, $r_e + 8 \leq m \leq s$ and $f_e$ with input $B_s(a_0)B_s(a_1)\ldots B_s(a_x)$ makes predictions on some set $D_s$ (used below) of strings of length up to $m^e$.

There are two cases:

(I) If such an $e$ is found, then let $B_{s+1}(a) = b$ if the function $f_e$ predicted $b$ at place $a$ and $a \in D_s$ and let $B_{s+1}(a) = B_s(a)$ otherwise. Furthermore, the $r_{e'}$ with $e' \geq e$ are updated to the new value $s^e$.

(II) If no such $e$ is found then let $B_{s+1} = B_s$ and do not change the restraints.

Let $C = \lim_{s\to\infty} B_s$. Whenever the set $D_s$ exists in stage $s$ and $m$ is defined as above, then $|y| \leq m^e$ for all $y \in D_s$. The conditions $e < \log(n)$ and $m \leq \max(U_n)$ enable to show the implications

$$n^{\log^k(n)} \geq m \Rightarrow n^{e\cdot\log^k(n)} \geq m^e \Rightarrow n^{\log^{k+1}(n)} > m^e.$$

From these implications it follows that for every single prediction of $f_{e'}$ with $e' \leq e$ the set $E$ of lengths of the places where predictions were made satisfies $|E \cap U_n| \leq 1$. It is easy to see that every requirement number $e$ acts at most $2^e$ times and thus, for any $x$, at most $2^{e+1}$ strings of the form $a_x 10^{m-1-|a_x|}$ with $m \in U_n$ and $n = |a_x| + 8$ are changed. So $x \in A$ iff the majority of the strings $a_x 10^{m-1-|a_x|}$ with $m \in U_n$ and $n = |a_x| + 8$

is in $C$.

This implies that $C$ is ptt-hard for **EXP** since the set $H = \{0^x : x \in A\}$ can be ptt-reduced to $C$. That this reduction is polynomial time computable follows from the fact that $|a_x| \leq \log(x)+2$ and so the largest query has length $n^{\log^{\log\log(n)}(n)}$ with $n \leq \log(x)+10$ which for almost all $x$ is below $x$ itself — note that $x$ and not $\log(x)+O(1)$ is the length of the input $0^x$.

It follows from the usual priority arguments, that the set $C$ is **Q**-generic as every requirement $e$ which can act infinitely often is eventually satisfied. Furthermore, including $r_e$ into the list of the updated restraints $r_{e'}$ with $e' > e$ makes sure that the requirement $e$ does not destruct its own work in later stages. □

**Theorem 4.6.** *There is a relativized world in which there is no Turing hard set for* **EXP** *which is general* **Q***-generic.*

**Proof.** Let $g(0) = 1$ and $g(n+1) = 2^{g(n)}$ for all $n$. Now the oracle $C$ is constructed such that it only contains strings of the form $0^{g(n+1)-g(n)}x$ where $n$ is a natural number and $x \in \{0,1\}^{g(n)}$. Furthermore, one chooses $C$ at length $g(n+1)$ such that the finite set

$$A_n = \{x \in \{0,1\}^{g(n)} : 0^{g(n+1)-g(n)}x \in C\}$$

differs for all $e \leq n$ from all sets decided by the $e$-th Turing reduction with time bound $2^{g(n)}-1$ relative to any oracle of the form $B \cup C$ where $B$ does not contain strings longer than $\frac{1}{n}g(n)$. Note that due to the time constraint the oracle $C$ is not queried at length $g(n+1)$ or beyond.

As there are less than $2^{g(n)/n+1}$ many strings in $B$ of length up to $g(n)/n$ and as there are only $n$ reductions considered, there are at most $n \cdot 2^{2^{g(n)/n+1}}$ many characteristic functions which $A_n$ should not take. But as $A_n$ can have $2^{2^n}$ many possible characteristic functions on the strings of length $n$, one can choose $A_n$ as desired.

The union $A$ of all $A_n$ is in **EXP** relative to the oracle $C$ by the choice of $C$: if $x$ does not have length $g(n)$ for any $n$ then $x \notin A$ and if $x$ has length $g(n)$ then we can compute, in exponential time, the string $0^{g(n+1)-g(n)}x$ and ask whether this string is in the oracle $C$.

Assume now, by way of contradiction, that $A$ is Turing reducible to a **Q**-generic set $B$ via $\varphi_e^B$ in time $p(n)$ where $p$ is a polynomial. Consider the **Q**-functional which predicts for every set $D$ on input $D(0)D(1)\ldots D(1^{g(n)/n-1})$ that all further values of $D$ up to $D(1^{p(g(n))})$ are 0 if $e < n \wedge p(g(n)) < g(n+1)$ and which is undefined otherwise. There are infinitely many predictions and so the set $B$ meets one of them, belonging to some $g(n)$. Now it follows that $\varphi_e$ computes the values $A_n(x)$ for all $x \in \{0,1\}^{g(n)}$ from the entries of the oracle $C$ of length $0, 1, \ldots, g(n)$ and from the entries of the set $B$ which have at most length $g(n)/n$. All other entries queried are 0 and can be ignored. Furthermore, $\varphi_e$ respects the time bound $2^{g(n)}-1$. As $n > e$, $A_n$ had been chosen previously such that this computation does not give $A_n$ and this contradiction gives that either $B$ is not Turing hard for **EXP** in the world relative to $C$ or that $B$ is not **Q**-generic. □

## 4.2 Approximability

We can ask how immunity notions for exponential time relate to the notions of approximability of sets as defined by Beigel, Kummer and Stephan [7] where a set is called *approximable* iff there is a **P**-function $f$ and a constant $k$ such that for every input $x_1, x_2, \ldots, x_k$ the function $f$ computes in polynomial time $k$ bits $y_1, y_2, \ldots, y_k$ such that one of these bits coincides with the characteristic function of $A$: $y_l = A(x_l)$ for some $l \in \{1, 2, \ldots, k\}$. We consider two special cases of approximability: a set $\{b_0, b_1, \ldots\}$ is **P**-*retraceable* iff there is a **P**-function $f$ with $f(a_x) = a_y$ for some $y \leq x$ and all natural numbers $x$ and $f(b_{n+1}) = b_n$ for all $n$. A **P**-*semirecursive set* is a set $B$ where one can compute from any finite set $D$ of strings in time polynomial in the sum of their lengths an input string a member $x \in D$ such that either $x \in B$ or $D \subseteq \overline{B}$. Note that **P**-retraceable and **P**-semirecursive sets are both approximable with the constant $k$ having the value 2.

**Theorem 4.7.** *An* **EXP***-avoiding set can be* **P***-retraceable but it is never* **P***-semirecursive.*

**Proof.** To see that an **EXP**-avoiding set can be **P**-retraceable, consider the retracing-function that maps every string $xy$ satisfying $2|x| - 1 \leq |xy| \leq 2|x|$ to $x$ on a set $A$ which contains the strings $x_0x_1x_2 \ldots x_n$ where $x_0 = 0$ and $x_{n+1}$ is defined inductively as a string in $\{0,1\}^m$ for which the conditional Kolmogorov complexity $K(x_{n+1}|x_0x_1x_2 \ldots x_n)$ is maximal with respect to computations needing time $2^{2^m}$. As the elements of sparse sets in **EXP** always have small double exponential Kolmogorov complexity, every sparse set in **EXP** contains only finitely many elements of $A$.

Let $B$ be any **P**-time semirecursive set. Then one can compute for any length $n$ in exponential time a string $x_n$ such that $x_n \in B$ whenever $B$ has any elements of length $n$. It follows directly that $B$ has an infinite intersection with the sparse set $E = \{x_0, x_1, x_2, \ldots\}$ whenever $B$ is infinite. □

Note that the second result also holds with "**EXP**-semirecursive" in place of "**P**-semirecursive".

The second result cannot be strengthened to **EXP**-hyperimmune sets. Dekker constructed a set $A$ which is hypersimple and semirecursive in the computability theoretic sense [18, Theorem II.6.16 and Theorem III.3.13]. An easy modification of the construction makes the set $A$ **P**-semirecursive. The complement $\overline{A}$ of $A$ is then also **P**-semirecursive. Furthermore, $\overline{A}$ is **EXP**-hyperimmune as $\overline{A}$ is already hyperimmune in the sense of computability theory.

## 4.3 Simplicity

**NEXP**-simple sets are those which are in **NEXP** and have an **NEXP**-immune complement. Such sets can be defined in a very general way.

**Definition 4.8 (Balcázar, Diaz, Gabarró [6]).** *Let* **CC** *be a class of sets. A set $A$ is called* **CC***-immune, if it is infinite and does not contain any infinite subset in* **CC***. A set $A$ is called* **CC***-simple, if $A$ is infinite, $A \in$ **CC** and the complement $\overline{A}$ is* **CC***-immune.*

Simplicity has been studied at many levels, ranging from computability [12, 18, 19, 20] to complexity theory [5, 14, 21, 23]. A major open problem is the existence of simple sets under some natural assumptions (maybe involving measure theory). Easy padding arguments give us the following relations between complexity-theoretic variants of simple sets. If there is a **NEXP**-simple set, then there is a **NE**-simple set (namely $\{x : x[1 \dots |x|^{1/k}] \in L\}$ where $L \in \mathbf{NTIME}(2^{n^k})$ is **NEXP**-simple). If there is an **NE**-simple set, then there is an **NP**-simple set in $\mathbf{P}/1$ ($\{x : |x| = y, y \in L\}$, where $L$ is **NE**-simple).

For any complexity class **CC** that allows us to run Ladner's delayed diagonalization technique we can show that any **CC**-simple degree bounds a **CC**-simple degree which is Turing-incomplete for **CC**. For example, any **NP**-simple degree bounds a degree Turing-incomplete for **NP**, which contains an **NP**-simple set (the same is true for **NEXP**). To prove this, we use Ladner's construction to construct a set $A \in P$ such that $S \cup A$ is Turing-incomplete for **CC**, and remains **CC**-simple. Of course $S \cup A \leq_T S$. It might be worth to try to exploit these ideas in order to attack the following open problem.

**Open Question 4.9.** *Is there a relativized world in which* **NEXP***-simple sets exist but none of them is Turing hard for* **NEXP***.*

# 5 Conclusion

The central topic of the paper is the question, for which reducibilities can **EXP**-immune, **EXP**-hyperimmune and **EXP**-avoiding sets be hard for **EXP**. With respect to many-one and conjunctive reducibilities, Theorem 3.1 shows that no **EXP**-immune set can be hard for **EXP**, which implies the same result for the more restrictive notions of **EXP**-hyperimmune and **EXP**-avoiding sets. With respect to disjunctive reducibility, Buhrman [10] constructed an **EXP**-immune set which is hard for **EXP**, while Theorem 3.3 states that there are no **EXP**-hyperimmune and thus also no **EXP**-avoiding sets which are hard for **EXP**. These results were obtained by using Theorem 2.7 which states that **EXP**-complete sets cannot be compressed conjunctively or disjunctively. Theorem 3.4 shows that there are no **EXP**-hyperimmune and thus also no **EXP**-avoiding sets which are parity-hard for **EXP**.

In the case of positive truth-table reducibility one has, on the one hand, a relativized world with an **EXP**-hyperimmune, even **EXP**-avoiding set which is ptt-hard for **EXP** while, on the other hand, Buhrman's result (Fact 3.5) implies that, for any $\alpha < \frac{1}{3}$ and in any relativized world there is no **EXP**-hyperimmune set which is hard for **EXP** with respect to truth-table reductions asking $n^\alpha$ many queries. Theorem 3.6 states that there is an **EXP**-avoiding Turing-hard set for **EXP** with respect to reductions using $\log^2(n)$ queries while as an immediate consequence of Fact 3.5 we know that no **EXP**-hyperimmune set is Turing-hard for **EXP** with respect to reductions using $\alpha \log(n)$ queries where $\alpha < \frac{1}{3}$. Finally, Theorem 4.5 shows that general **Q**-generic sets can be positive truth-table hard for **EXP** in those relativized worlds where $\mathbf{EXP} \subseteq \mathbf{P/poly}$ while there are, by Theorem 4.6, other relativized worlds where general **Q**-generic sets are even not Turing-hard for **EXP**.

**Acknowledgments.** We would like to thank Stephen Fenner for discussions and Wolfgang Merkle for important comments.

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