Removing Even Crossings

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Abstract

An edge in a drawing of a graph is called *even* if it intersects every other edge of the graph an even number of times. Pach and Tóth proved that a graph can always be redrawn so that its even edges are not involved in any intersections. We give a new and significantly simpler proof of the stronger statement that the redrawing can be done in such a way that no new odd intersections are introduced. We include two applications of this strengthened result: an easy proof of a theorem of Hanani and Tutte (the only proof we know of not to use Kuratowski's theorem), and the new result that the odd crossing number of a graph equals the crossing number of the graph for values of at most 3. The paper begins with a disarmingly simple proof of a weak (but standard) version of the theorem by Hanani and Tutte.

1 The Hanani-Tutte Theorem

In 1970 Tutte published his paper "Toward a Theory of Crossing Numbers" [16] containing the following beautiful theorem.

> In any planar drawing of a non-planar graph there are two non-adjacent edges that cross an odd number of times. In other words: if a graph can be drawn such that every pair of non-adjacent edges intersects an even number of times, then the graph is planar.

Tutte acknowledges earlier proofs of the same result, including the paper "Über wesentlich unplättbare Kurven im drei-dimensionalen Raume" [5] published in 1934 by Chaim Chojnacki (who later changed his name to Haim Hanani). While there is general agreement that the result itself is "remarkable" [12, 3], and "nice" [1], the same cannot be said of its proofs. Both Hanani and Tutte took the same general approach in their proofs using Kuratowski's theorem: If the graph is non-planar it contains a subdivision of $K_{3,3}$ or K_5, so they only have to show that any drawing of *these* graphs contains two non-adjacent edges that cross an odd number of times. Hanani opts for a more topological approach, while Tutte develops an algebraic theory of crossing numbers.

We want to present a very intuitive and entirely geometric proof of the result which, furthermore, does not use Kuratowski's theorem. We begin by proving a slightly weaker result.

Let us call an edge in a drawing *even* if it intersects every other edge an even number of times.[1]

Theorem 1.1 (Hanani-Tutte, weak version) *If G can be drawn in the plane so that all its edges are even, then G is planar.*

Proof We may assume that G is connected, since components may be redrawn arbitrarily far apart. Fix a plane drawing D of G in which every pair of edges intersects an even number of times. We prove the result by induction on the number of edges in G. To make the inductive step work, we keep track of the *rotation* of each vertex, that is, the cyclic order in which edges leave the vertex in the drawing. The mapping from the vertices of G to their rotations is called the *rotation system* of D. We will prove the following stronger statement:

> If D is a drawing of a multigraph G so that any pair of edges intersects an even number of times in D, then G is planar and can be drawn without changing the rotation system.

We begin with the inductive step: if there are at least two vertices in G, then there is an even edge $e = uv$. Pull v towards u as shown in the left part of Figure 1.

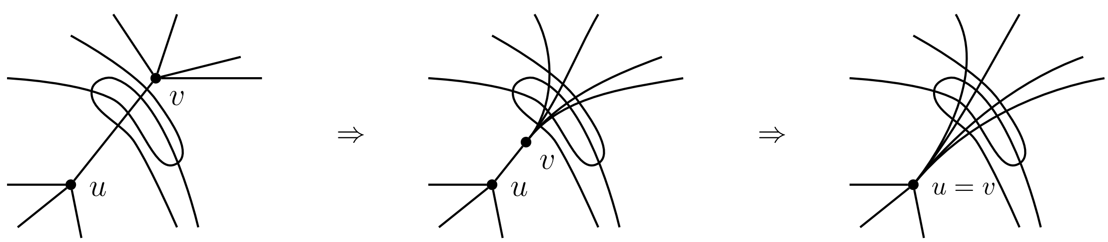

Figure 1: Pulling an endpoint (left) and contracting the edge (right).

Since e was an even edge, the edges incident to v remain even. The pulling move will introduce self-intersections in curves that intersect e and are adjacent to v. Since drawings are typically defined not to have self-intersections, we remove them by using the move shown in Figure 2 (although we could preserve self-intersections and instead modify the analysis slightly).

Now that uv no longer has any intersections, we contract it to obtain a new graph G' in which the rotations of u and v are combined appropriately (see the right part of Figure 1).

[1] We make the standard assumptions on the drawing of a graph, in particular we assume that any pair of edges intersects only finitely often [11, page 230].

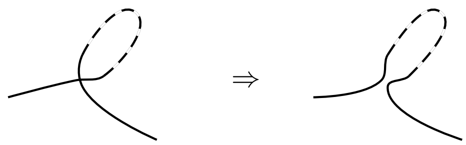

Figure 2: Removing a self-intersection.

By the inductive assumption, there is a planar drawing of G' respecting the rotation system. In such a drawing, we can simply split the vertex corresponding to u and v, reintroducing the edge e between them without any intersections. Hence G is planar respecting the rotations of all its vertices.

If G contains only a single vertex, then it might have several loops attached to it. Since all the loops in G are even, it cannot happen that we find edges leaving in order a, b, a, b since this would force an odd number of intersections between a and b. Hence, if we consider the regions enclosed within the two loops in a small enough neighborhood of the vertex, either they are disjoint or one region contains the other. Then it is easy to show that there must be a loop whose ends are consecutive in the rotation system. Removing this edge we obtain a smaller graph G' which, by inductive assumption, can be drawn without intersections and with the same rotation system. We can then reinsert the missing loop at the right location in the rotation system by making it small enough.

In the base case, we simply draw a single vertex with no edges. □

We can restate the result in terms of crossing numbers. The *crossing number* of a drawing of a graph is the total number of crossings of each pair of edges. The *crossing number of* G, $\mathrm{cr}(G)$, is the smallest crossing number of any drawing of G. The *odd crossing number* of a drawing is the number of pairs of edges that cross an odd number of times. The *odd crossing number of* G, $\mathrm{ocr}(G)$, is the smallest odd crossing number of any drawing of G. It follows from the definition that

$$\mathrm{ocr}(G) \leq \mathrm{cr}(G).$$

Theorem 1.1 shows that $\mathrm{ocr}(G) = 0$ implies $\mathrm{cr}(G) = 0$ (that is, G is planar). The original result by Hanani and Tutte draws the same conclusion under the weaker assumption that all pairs of *non-adjacent* edges intersect an even number of times. This suggests the concept of the *independent odd crossing number*, $\mathrm{iocr}(G)$, as the smallest number of pairs of non-adjacent edges of G that intersect an odd number of times in any drawing of G. The original Hanani-Tutte result [5, 16] can then be stated as follows.

Theorem 1.2 (Hanani-Tutte, strong version) *If* $\mathrm{iocr}(G) = 0$, *then* $\mathrm{cr}(G) = 0$.

We will give a proof of the strong version in Section 3.1. As far as we know this is the first direct and geometric proof of the theorem, not making use of Kuratowski's theorem.

Remark 1 We include a short survey of previous proofs of both the weak and the strong version of the Hanani-Tutte theorem. Let us begin with proofs of the strong version. Two papers in 1976, one by Kleitman [8], the other by Harborth [6] showed that the parity of $\text{iocr}(G)$ is independent of the drawing of G if G is either K_{2j+1} or $K_{2j+1,2j+1}$. Norine [9] supplies a different proof of this result and observes that it implies the strong version of the Hanani-Tutte theorem by an application of Kuratowski's theorem. Székely [15] shows that $\text{iocr}(K_{3,3}) = \text{iocr}(K_5) = 1$ simplifying Tutte's algebraic approach. Again, an application of Kuratowski's theorem yields the strong version of the Hanani-Tutte theorem.

There are several proofs of the weak version, typically as corollaries of more general results. Pach and Tóth [12] showed that intersections along even edges can be removed even in the presence of edges that are not even. In Section 2 we will show how to obtain a stronger version of their result using our methods. There also is a proof by Cairns and Nikolayevsky [4, Lemma 3] using homology which shows that the weak version is true on surfaces of any genus.

Before moving on to the next section, let us have another look at the proof of Theorem 1.1. All we need to make its inductive argument work is a spanning tree of even edges; let us call such a spanning tree *even*. If we contract along the edges of that spanning tree, we obtain a single vertex with a bouquet of loops, some intersecting oddly, some evenly. However, whether two loops intersect oddly or evenly only depends on whether their endpoints in the rotation system interleave or not, and it is easy to redraw a single vertex and its loops so that pairs of loops that intersect evenly do not intersect at all, and pairs of loops intersecting oddly, intersect once. Since contracting along even edges does not change the parity of the number of intersections between edges, we have just shown: If G has a drawing realizing $\text{ocr}(G)$ that contains an even spanning tree, then $\text{ocr}(G) = \text{cr}(G)$.

Theorem 1.3 *If* $2\,\text{ocr}(G) < \lambda(G)$*, where* $\lambda(G)$ *is the edge-connectivity of* G*, then* $\text{ocr}(G) = \text{cr}(G)$*.*

Proof Fix a drawing of G realizing $\text{ocr}(G)$. All we have to show is that in that drawing G contains an even spanning tree. We can assume that G is connected, and build a spanning tree T iteratively, starting with an arbitrary vertex. As long as T does not span G yet, there are always at least $\lambda(G) \geq 2\,\text{ocr}(G) + 1$ edges connecting T and $G - T$. Since at most $2\,\text{ocr}(G)$ of these can be involved in an odd crossing, at least one of these edges is even. □

What about the independent odd crossing number? It seems we need a stronger assumption than the existence of an even spanning tree. Call a drawing *evenly* 2*-connected* if its subgraph of even edges is spanning and 2-connected. Note that a drawing is evenly 2-connected if and only if it contains an even and spanning 2-connected subgraph.

Lemma 1.4 *If* G *has a drawing realizing* $\text{iocr}(G)$ *which is evenly* 2*-connected, then* $\text{iocr}(G) = \text{ocr}(G) = \text{cr}(G)$*.*

Proof Fix an evenly 2-connected drawing of G realizing $\text{iocr}(G)$. Consider an arbitrary vertex v, and a disk-shaped neighborhood of v in which none of the edges leaving v intersect (other than at v). Erase the area of the disk, and create a mirror-copy of the rest of the drawing inside the disk by circular inversion. As a result of this operation, all edges intersecting the boundary of the disk intersect an even number of times. Since G is evenly 2-connected, $G - v$ contains an even spanning tree, and we can contract the mirror-copy of that spanning tree within the disk to a single vertex, which we call v (deleting all resulting loops). The edges leaving v now intersect each other evenly. Repeating this process for all vertices yields a drawing for which $\mathrm{iocr}(G) = \mathrm{ocr}(G)$. Since the final drawing is still evenly 2-connected, and is minimal with respect to $\mathrm{ocr}(G)$ we can, as before, conclude that $\mathrm{ocr}(G) = \mathrm{cr}(G)$. □

2 Removing Even Crossings

Pach and Tóth [12, Theorem 1] generalized the weak version of the Hanani-Tutte theorem by showing that one can always redraw even edges without crossings—even in the presence of odd edges. Their proof is a nontrivial extension of Tutte's and Hanani's approach of extending Kuratowski's theorem. We show that our inductive approach gives a much simpler proof of the Pach-Tóth result. In fact, it yields the stronger conclusion that we can perform the redrawing without adding pairs of edges that intersect an odd number of times; in particular the odd crossing number does not increase. We give two applications of our strengthened result in Section 3.

Theorem 2.1 *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.*

Proof We assume without loss of generality that G is connected. Fix the plane drawing D of G, and let E_0 be the set of even edges in D. We prove the result by induction on the number of even edges in the drawing. To make the inductive step work, we keep track of the rotation of each vertex. We will prove the following stronger statement:

> If D is a drawing of a multigraph G with even edges E_0, then there is a drawing D' of G in which none of the edges in E_0 intersect, D' and D have the same rotation system and there are no new pairs of edges that intersect an odd number of times.

As in the proof of Theorem 1.1, we contract an even edge uv (with $u \neq v$) to obtain G'. Observe that this does not lead to any new odd intersections, since the contraction does not affect whether a pair of edges intersects an odd number of times. By the inductive assumption, there is a planar drawing of G' respecting the rotations of the vertices, which does not introduce any new pairs of edges intersecting an odd number of times. In such a drawing, we split the vertex corresponding to u and v, reintroducing the edge uv so that it does not intersect any edge. Thus we obtain a drawing of G that respects the rotation of every vertex, and there are no new pairs of edges that intersect an odd number of times.

In this proof, the resulting base case is more complex than in Theorem 1.1: we have a drawing D of a multigraph G all of whose even edges are loops. In other words any edge between two distinct vertices is involved in an odd intersection with some other edge.

Pick an even edge e with endpoint v, and consider the region enclosed by the loop formed by e. Edges whose endpoints are both in the region and loops at v whose ends approach v from inside the region are called *e-inside*. If both endpoints of an edge lie outside the region, or if a loop at v approaches its ends from outside the region, we call the edge *e-outside.* Since e is even, every edge other than e is either e-inside or e-outside.

We first focus exclusively on redrawing the e-inside edges, leaving the e-outside edges untouched. Draw a small loop ℓ at v that does not intersect any e-inside edges and lies outside the region enclosed by e as shown in Figure 3.

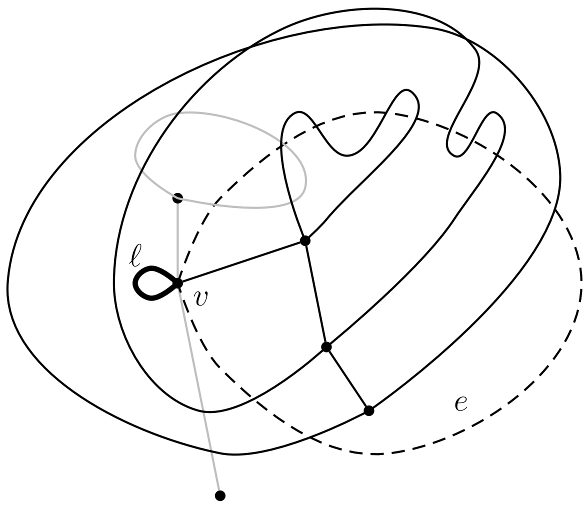

Figure 3: Even edge e and loop ℓ at v. (The gray lines are e-outside.)

Think of the drawing as being on a sphere. Then the loop ℓ bounds two disks, one of which contains all the e-inside edges. That region can be continuously deformed to the region enclosed by e, so that ℓ goes to e and v remains fixed. As a result every e-inside edge is properly contained in the region enclosed by e, while the e-outside edges are unchanged (see Figure 4).

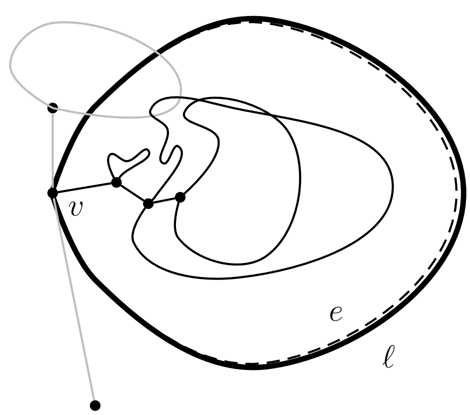

Figure 4: After the deformation.

Now we shrink e and the e-inside edges enclosed by e towards v until the region enclosed by e no longer intersects any e-outside edges. (This is possible because in a small enough neighborhood of v, the e-outside edges appear to be nearly straight lines arranged according to the rotation at v.) At this point, the edge e does not intersect any other edges in the drawing; furthermore, the rotation system remained the same, and we did not introduce any odd intersections (though we might have removed some). This concludes the proof. □

3 Applications

3.1 The Hanani-Tutte Theorem, Revisited

In this section we prove the strong version of the Hanani-Tutte theorem without recourse to Kuratowski's theorem. Our main tool will be Theorem 2.1. Note that the independent odd crossing number of the graph does not increase when applying Theorem 2.1, a fact we need below.

Theorem 3.1 (Hanani-Tutte, strong version) *If* $\mathrm{iocr}(G) = 0$ *then* $\mathrm{cr}(G) = 0$.

Proof The core idea of the proof is to locate cycles in the graph, and, for each cycle, to make its edges even and then to redraw the cycle without intersections, by applying Theorem 2.1. We say that such edges have been *processed*. However, a straightforward induction over the number of cycles in G consisting of even edges causes problems when changing the rotation at a vertex and when modifying G by splitting vertices. Hence, the overall induction will be over the *weight*

$$w(G) := \sum_{v \in V} d(v)^3,$$

where V is the vertex set of G and $d(v)$ the degree of v in G. For two graphs with the same weight, the induction will be over the number of *unprocessed* edges, where initially all edges of G are unprocessed. Every *processed* edge will be even; in fact, a processed edge will have no intersections. Also, a processed edge always belongs to a cycle of processed edges.

We begin with a drawing of G witnessing $\mathrm{iocr}(G) = 0$. If all edges of the drawing are even, we are done, since then the graph is planar (by either Theorem 1.1 or Theorem 2.1). Therefore, there is at least one odd edge, and this edge is necessarily unprocessed. Pick such an edge e. There are two possibilities: $e = uv$ is a cut-edge, or it is contained in a cycle. If e is a cut-edge, we can remove it, yielding two smaller graphs G_1 and G_2. By induction on the weight, both are planar. Moreover, both have planar embeddings so that u and v are on the outer face. Hence, we can draw e connecting u and v to obtain a planar drawing of G.

Therefore, we may assume that e lies on a cycle C. First, let us consider the case that for every vertex u of C, either every two edges incident at u intersect evenly, or every edge incident to u is unprocessed. In the latter case, we can modify the rotation at u by redrawing G in a small neighborhood of u so that the two edges of C incident at u intersect evenly with each other (if necessary). We can then modify the rotation of the remaining edges incident at u so each of them intersects both the edges of C incident at u evenly. Thus, we can assume that the two edges of C incident to u are even.

By repeating the redrawing as necessary for all vertices on C, we obtain a drawing of G in which all edges of C are even—without changing $\mathrm{iocr}(G)$ or adding intersections to processed edges. Applying Theorem 2.1 yields a drawing of G in which the edges of C are free of intersections: they have been *processed*. Since all previously processed edges are even, they are also drawn without intersections, as required. Since the number of unprocessed edges decreases, we may apply induction to obtain a planar drawing of the graph.

Otherwise, for some vertex u of C, u is incident to a processed edge and there are two edges incident to u that intersect an odd number of times. The processed edge is contained in a cycle C' of processed edges, which is drawn without crossings, and thus divides the plane into two regions. The two oddly-intersecting edges must be in the same region. Split the vertex u into two adjacent vertices v and w, with edges incident to u in that region made incident to v, and the other edges incident to u (including those incident to u on C') made incident to w (see Figure 5).

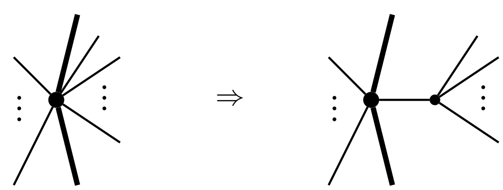

Figure 5: Splitting the vertex. (The thick edges belong to C.)

Since there are no intersections between the groups, the splitting does not increase the independent odd intersection number. Since the degrees of v and w are each at least 3 and $d(u) = d(v) + d(w) - 2$, $d(v)^3 + d(w)^3 < d(u)^3$, as desired. Hence, the new graph has a planar drawing by induction, and vw can be contracted to obtain a planar drawing of G. □

3.2 Small Crossing Numbers

When applying Theorem 2.1 to draw conclusions about the odd crossing number, we proceed as follows: Draw G to minimize the odd crossing number $\mathrm{ocr}(G)$ (call pairs that intersect an odd number of times *odd pairs*, edges are belonging to an odd pair are *odd*; the remaining edges are *even*). Using Theorem 2.1, we can redraw the even edges so they are not involved in any intersections and so that the new drawing still has odd crossing number $\mathrm{ocr}(G)$. Now the even edges form a plane graph G' and each odd edge lies entirely within some face of G'. We can now process the odd edges within each face separately to obtain results on crossing numbers.

Example 3.2 For example, let us reconstruct the argument by Pach and Tóth which shows that $\mathrm{cr}(G) \leq 2\,\mathrm{ocr}(G)^2$. Consider a subgraph H of G drawn in the plane consisting of the (odd) edges within a face of G' and the (even) edges on the boundary of the face. We can redraw the odd edges of H so that each pair intersects at most once; after the redrawing we have at most $\binom{|E(H)|}{2}$ crossings within that face of G'. If we do the same for every face of G' we can conclude that $\mathrm{cr}(G)$ is at most the sum of $\binom{|E(H)|}{2}$ where H ranges over the odd subgraphs in the faces of G'. Furthermore, we know that there are at most $2\,\mathrm{ocr}(G)$ odd edges in total, hence $\mathrm{cr}(G) \leq \binom{2\,\mathrm{ocr}(G)}{2} \leq 2\,\mathrm{ocr}(G)^2$. ■

This plan is also useful when the odd crossing number is quite small, though not necessarily zero. Recall that by definition $\mathrm{ocr}(G) \leq \mathrm{cr}(G)$, and that by the result of Hanani and Tutte, $\mathrm{ocr}(G) = 0$ implies $\mathrm{cr}(G) = 0$. This suggests that perhaps $\mathrm{ocr}(G) = \mathrm{cr}(G)$ for all graphs G (see [10, 17, 1]). While this conjecture turns out to be false [13], we can show, using our approach, that $\mathrm{ocr}(G) = \mathrm{cr}(G)$ if $\mathrm{ocr}(G)$ is small enough.

As we explained above, let G' be the subgraph of G consisting of the even edges, and fix a drawing of G in which G' is planar. We consider each face separately: suppose that H is a subgraph drawn in the plane consisting of the (odd) edges within a face of G' and the (even) edges on the face boundary. We will show that if there are only a few odd crossings in H, then H can be redrawn with exactly $\mathrm{ocr}(H)$ crossings, with each component of the face boundary either having its embedding unchanged or having its embedding "flipped" in the plane. If we sequentially process faces in this way and agree that when a face boundary is flipped, so is the rest of the drawing currently on that side of the boundary, then the entire graph is redrawn with $\mathrm{ocr}(G)$ crossings, as desired.

In particular, we will obtain the following theorem.

Theorem 3.3 *If G is a graph with $\mathrm{ocr}(G) \leq 3$, then $\mathrm{ocr}(G) = \mathrm{cr}(G)$.*

Interestingly, this mirrors a result for the *rectilinear crossing number*, $\mathrm{rcr}(G)$, which is the smallest number of crossings in a straight-line drawing of G: Bienstock and Dean [2] showed that $\mathrm{rcr}(G) = \mathrm{cr}(G)$ for graphs G with $\mathrm{cr}(G) \leq 3$. They also constructed graphs G with $\mathrm{cr}(G) = 4$ and $\mathrm{rcr}(G)$ arbitrarily large.

The previous discussion reduces the proof of the theorem to the following lemma.

Lemma 3.4 *Let H be a multigraph, and let B be the subgraph consisting of its even edges and all their endpoints. Suppose that H is embedded in the plane so that B is a plane graph, a single face F of B contains every odd edge of H (except for endpoints), and every edge of B is on the boundary of F. (Then the boundary of F is B.)*

Let $\mathrm{op}(H)$ be the number of odd pairs. If $\mathrm{op}(H) \leq 3$, then the odd edges may be redrawn so that $\mathrm{op}(H)$ decreases or the number of crossings equals $\mathrm{op}(H)$.

Proof We may assume that H is connected, since otherwise we translate components to be far apart and deal with each component separately.

Contract an even edge uv that is not a loop, combining the rotations of u and v appropriately to get a rotation for the new vertex, and adjusting the drawing of H appropriately. Repeat, until each component of F is a bouquet of loops, each with an empty interior (except at most one, which would be the boundary of the outer face and have empty exterior). Delete all such loops. If we now redraw edges, preserving the rotation at each vertex, then the loops may be redrawn and the vertices can be split, recovering the original drawing of F with the original rotation system, without introducing any new crossings.

We are now considering a connected multigraph G drawn so that every edge is involved in an odd crossing, with $\mathrm{op}(H)$ pairs of edges that cross an odd number of times. Note that the ends of a loop cannot appear consecutively in a rotation since it could be redrawn with no crossings.

If G has only one vertex, then, as before, the loops can be redrawn with only $\mathrm{op}(H)$ crossings while preserving the rotation of the vertex. Thus we may assume that G has at least two vertices.

If v is incident to only one nonloop edge uv, we move v to be very close to u, redrawing uv so as not to intersect any other edge. We can then draw each loop e at v so that (1) e is small enough to not cross any edge other than those incident to v, (2) e does not cross uv, and (3) e crosses each loop at v at most once, and only crosses if the ends of the two loops alternate in the rotation at v. Since the loops at v can be drawn with no fewer odd crossings, and since uv no longer crosses any edge an odd number of times, $\mathrm{op}(H)$ decreases as desired.

Now suppose that v is incident to exactly two nonloop edges $e = uv$ and $e' = vw$. Temporarily ignore the loops at v. The concatenation of the drawings of e and e' form a curve, along which we move v until v is close enough to u so that the modified drawing of e crosses no other edge. We can then remove any crossings from the modified e' as in Figure 2; note that if uw crosses another edge an odd number of times then either uv or vw used to cross that edge an odd number of times. Thus, $\mathrm{op}(H)$ must decrease, ignoring contributions from loops at v. Now we show that we can also draw the loops at v so as not to increase their contribution to the number of odd crossings from the original drawing.

If v is a cut-vertex, then we can draw the loops at v large enough so that they intersect nothing but one another; also they can be drawn so that two loops at v intersect at most once, and only intersect when forced to by the rotation at v. Otherwise, there is a nonloop cycle through v. Temporarily ignore everything but v, the loops at v, and an additional loop placed in the rotation at v where the nonloop cycle goes. We can draw the bouquet of loops optimally (so that number of odd crossings equals number of crossings equals number of loops whose ends alternate in the rotation) with the special loop drawn much larger than the others. Then we replace the special loop by the previously ignored portion of the graph drawing. The odd crossings in the finished drawing are the (disjoint) union of such crossings obtained in the two steps we used. Note that the loops are not involved in any more odd crossings than was forced by the rotation at v. Therefore, in either case, $\mathrm{op}(H)$ decreases overall.

Thus, each vertex must be incident to at least three nonloop edges.

Next, consider any vertex v of degree 3. If one edge e incident to v has an odd number of crossings with each other edge incident to v, then we can add a twist to e near v so that e crosses each of the other edges exactly once more. This lowers the number of odd crossings by 2. If that does not occur, then at most one pair of edges incident to v cross an odd number of times. Suppose that uv and vw are two such edges. Then we can flip the rotation at v and redraw these edges in a small neighborhood of v to add exactly one crossing between them; this lowers the odd crossing number by exactly one. This shows that for any vertex of degree 3, no two of its incident edges form an odd pair.

Next, suppose that there is a counterexample with exactly 2 vertices, and choose one with a minimum number of edges. We show that it has no loops. It helps to switch the viewpoint from vertices with a rotation system to a map on the annulus: Consider a map on the annulus for which $\mathrm{ocr}(M) \neq \mathrm{cr}(M)$ and a drawing with $\mathrm{ocr}(M)$ odd crossings and more than $\mathrm{ocr}(M)$ crossings. A loop is then an edge e whose endpoints are on the same boundary components of the annulus. Observe that the number of odd crossings that e

makes depends only on the homotopy class of the drawing. Therefore, given a drawing of $M - e$ with $\operatorname{ocr}(M - e)$ odd crossings, adding e yields a drawing of M with $\operatorname{ocr}(M)$ odd crossings as long as e is drawn in a certain (fixed) homotopy class. By assumption, there is a drawing of $M - e$ with $\operatorname{ocr}(M - e)$ crossings. We can draw e in a given homotopy class so that it runs alongside the boundary of the annulus, ensuring that it makes the same number of crossings as odd crossings. This gives us a drawing of M with $\operatorname{ocr}(M)$ crossings, a contradiction. It follows that in order to show that for all maps M on the annulus with $\operatorname{ocr}(M) \leq 3$ we must have $\operatorname{ocr}(M) = \operatorname{cr}(M)$, it suffices to consider maps with no loops.

Suppose that $\operatorname{op}(H) \leq 2$. Since every current edge is part of one of the $\operatorname{op}(H)$ odd pairs, there are at most $2\operatorname{op}(H)$ edges—which is at most 4. Then the degree sum is at most 8, so there are less than 3 vertices; in fact there are exactly 2 vertices, say, u and v. If there are 4 edges, then they are partitioned into two odd pairs. There cannot be only 3 edges incident to a vertex since they form no odd pairs, and one other edge cannot create two odd pairs. So we may assume that there are 4 edges from u to v and no loops. Label the rotation at u clockwise to be $1, 2, 3, 4$; this determines the labeling at v, since the rotation system is given. Since the rotation at v may be flipped, we have 3 cases, depending on what label is opposite of 1 at v: If it is 3, then we can redraw with no crossings. If it is 4, then we can draw edges 1 and 4 so that neither are involved in crossings, and edges 2 and 3 cross exactly once. If 2 is opposite 4 at v, then we can draw so that edges 1 and 4 cross once, and edges 2 and 3 cross once, and there are no other crossings.

We may now assume that $\operatorname{op}(H) = 3$. The number of edges is at most 6, so the degree sum is at most 12, and there are at most 4 vertices. In particular, if there are 4 vertices, then the graph is 3-regular. If one vertex v is adjacent to the other three, then two of them must be adjacent to each other. No matter what the rotation is at the other vertex, the graph can be drawn with at most 2 crossings. Otherwise the graph can only be a 4-cycle with two non-adjacent edges doubled; this can be drawn with at most 2 crossings as well.

Suppose that there are 3 vertices now. There are either 5 or 6 edges in this case. If there is no edge between two vertices u and w, then each must have three edges to the other vertex v. The three edges from u to v and one edge from v to w can be drawn with no crossings, then each of the other two edges can be added creating at most one crossing each. Thus we may assume that the graph contains a triangle. Since each vertex is incident to at least three nonloops, there are at least 2 doubled edges. There cannot be just these edges, since the single edge would not cross any other edge. The edges at v can be drawn to contribute at most one crossing, and a sixth edge can be drawn to contribute at most 2 crossings no matter what the rotation system is, so $\operatorname{cr}(H) \leq 3$ as desired.

Now we may assume that there are exactly two vertices, u and v. It will be convenient to think of this graphs as a map on the annulus again; as proved earlier, we may assume that there are no loops. No matter what the rotations are, 4 edges can be drawn with at most one crossing, and a fifth edge can be added with at most 2 more crossings. Thus we can assume that there are 6 edges, and a drawing in which they are partitioned into 3 odd pairs. Fix the rotation flip that yields this. Then for any single edge, the parity of the number of twists is the sole factor that determines which edges it crosses an odd number of times. Thus, if we pick three edges that form no odd pairs, we may assume that they form no crossings at all. Sequentially add the other three edges; in order to cross only its partner an odd number of times, the rotations must be equivalent to $(1, 2, 3, 4, 5, 6)$, $(2, 1, 4, 3, 6, 5)$, which can be drawn with exactly 3 crossings. □

4 Conclusion

The core result of this paper, Theorem 2.1, shows that we can remove even crossings without introducing any odd crossings. With this tool at hand, we were able to give new and entirely geometric proofs of the Hanani-Tutte theorem and the Pach-Tóth result that $\mathrm{cr}(G) \leq 2\,\mathrm{ocr}(G)^2$. We believe that this strengthened form is crucial for obtaining improved upper bounds on $\mathrm{cr}(G)$ in terms of $\mathrm{ocr}(G)$. As evidence we offer our proof of the equality of $\mathrm{cr}(G)$ and $\mathrm{ocr}(G)$ up to values of 3 (even the question whether $\mathrm{ocr}(G) = 1$ implies $\mathrm{cr}(G) = 1$ had previously been open). We can also apply the theorem to show that every graph G has a drawing D with $\mathrm{ocr}(G) = \mathrm{ocr}(D)$ and $\mathrm{cr}(D) \leq 9^{\mathrm{ocr}(G)+1}$, a result which has consequences for the parameterized complexity of computing the odd crossing number [14].

The method of contracting edges in graphs with rotation systems bears further investigation. While it is not a new idea—for example, it is used to prove Fáry's theorem that every planar graph has a straight-line drawing—we believe it to be a fruitful, intuitive idea that has not yet achieved its potential and allows many further variations. One can, for example, apply it to edges that are not even. Using this approach, we can show that

$$\mathrm{cr}(G) \leq |V(G)|^4\,\mathrm{ocr}(G)$$

for any multigraph G [13]. For multigraphs with a fixed number of vertices it shows that there is at most a linear gap between crossing number and odd crossing number.

Finally, since our proof of the Hanani-Tutte theorem does not use Kuratowski's theorem, it offers the possibility of finding a purely topological proof of Kuratowski's theorem. We were recently informed by Hein van der Holst that he has found such a proof [7].

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